

A characterization of certain families of
well-covered circulant graphs

by

Rania Moussi

A Thesis Submitted to
Saint Mary's University, Halifax, Nova Scotia
in Partial Fulfillment of the Requirements for
the Degree of Master of Science in Applied Science.

October, 2012, Halifax, Nova Scotia

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Date: October 5, 2012

Abstract

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Abstract: A graph G is said to be *well-covered* if every maximal independent set is a maximum independent set. The concept of well-coveredness is of interest due to the fact that determining the independence number of an arbitrary graph is *NP*-complete, and yet for a well-covered graph it can be established simply by finding any one maximal independent set.

A *circulant graph* $C(n, S)$ is defined for a positive integer n and a subset S of the integers $1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, called the connections. The vertex set is \mathbb{Z}_n , the integers modulo n . There is an edge joining two vertices j and i if and only if the difference $|j - i|$ is in the set S . In this thesis, we investigate various families of circulant graphs. Though the recognition problem for well-covered circulant graphs is *co-NP*-complete, we are able to determine some general properties regarding these families and to obtain a characterization.

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List of Symbols

In this list G is a graph and T is a finite subset of the integers (mod n).

$C(n, S)$ The circulant graph on n vertices with generating set S

$E(G)$ The edge set of G

\overline{G} The complement of G

$|G|$ The cardinality of G

$G[V(\mathcal{U})]$ The subgraph of G induced by $V(\mathcal{U})$

$K_{m,n}$ The complete bipartite graph of size m, n

K_n The complete graph on n vertices

$N_G(v)$ The open neighborhood of a vertex v in G

$N_G[v]$ The closed neighborhood of a vertex v in G

$V(G)$ The vertex set of G

$\beta(G)$ The independence number of G

$\alpha(G)$ The covering number of G

$G \cong H$ The symbol for G is isomorphic to H

\mathbb{Z} The set of integers

\mathbb{Z}_n The group of integers (mod n)

$-T$ $-T = \{y \in \mathbb{Z}_n : y + t \equiv 0 \text{ for some } t \in T\}$

$\langle T \rangle$ $\langle T \rangle = T \cup (-T)$

$u \sim v$ The symbol for u is adjacent to v

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Chapter 1

Introduction

A graph G is said to be *well-covered* if every maximal independent set is a maximum independent set. This notion, introduced by M. D. Plummer in 1970 [29], was originally defined by him in terms of point cover, thus the name well-covered. However, the independent set point of view has become predominant among researchers. Note that a *point cover* is a set of points (vertices) such that every edge is incident with some point in the cover [8] and an *independent set* (*stable set*) is a set of vertices such that no two members are joined by an edge. A *maximum independent set* is the largest independent set for a given graph. An independent set is a *maximal independent set* if it is not a proper subset of another independent set. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\beta(G)$ and the number of vertices in a minimum covering of G is the *covering number* of G and is denoted by $\alpha(G)$. Furthermore, it is well-known that the complement of a point cover is an independent set. Thus, one could choose either point of view to tackle the property of well-coveredness.

The concept of well-coveredness captured Plummer's attention mainly due to the fact that determining the independence number of an arbitrary graph is *NP*-complete, and yet for a well-covered graph it can be established simply by finding any one maximal independent set. Note that to determine whether an arbitrary graph is well-covered, no method is known which is significantly faster than comparing all maximal independent sets; whereas, determining non-well-coveredness is a simpler process; it

suffices to find two maximal independent sets of differing cardinality.

Since the introduction of this class of graphs, much research has been conducted and various results have been published. Even though the problem of determining well-coveredness is *co-NP*-complete, various subclasses of well-covered graphs have been characterized. Furthermore, a number of these families are recognizable in polynomial time. For an in-depth review the reader is referred to the survey papers by Plummer [30] and Hartnell [8].

A *circulant graph* $C(n, S)$ is defined for a positive integer n and a subset S of the integers $1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, called the connections. The vertex set is \mathbb{Z}_n , the integers modulo n . There is an edge joining two vertices j and i if and only if the difference $|j - i|$ is in the set S . The problem of determining whether a circulant graph is well-covered was investigated by R. Hoshino [33] in his 2007 dissertation. Among other things he showed that the recognition problem for well-covered circulant graphs is *co-NP*-complete. Though it seems that there are many complex issues when it comes to recognizing whether an arbitrary circulant graph is well-covered, several characterizations of subclasses of these graphs have been attained so far. In addition, many techniques are used to produce new graphs from existing ones and using graph products is a well known approach.

1.1 Basic Terminology

A *graph* G consists of a finite nonempty set $V(G)$ of *vertices* and a set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. The *vertex set* of G is denoted by $V(G)$ and the *edge set* is denoted by $E(G)$. For simplicity the edge $e = \{u, v\}$ is denoted by uv or vu . If $e = uv$ is an edge of a graph G , then u and v are *adjacent vertices*, denoted by $u \sim v$. The vertices u and v are referred to as *neighbors* of each other. In this case, the vertex u and the edge e (as well as v and e) are said to be *incident* with

each other. Distinct edges of G incident with a common vertex are *adjacent edges*. Furthermore, the (*open*) *neighborhood* of a vertex v in a graph G , denoted $N(v)$, is the set of vertices adjacent to v , and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$.

The *order* of a graph G is the cardinality of its vertex set and is denoted by $|V(G)|$; while the *size* of a graph G is the cardinality of its edge set and is denoted by $|E(G)|$. The *degree of a vertex* v in a graph G is the number of edges of G incident with v and is denoted by $\deg(v)$. A vertex of degree 0 in G is referred to as an *isolated vertex*.

Two graphs G_1 and G_2 are *isomorphic* if there exists a one-to-one correspondence ϕ from $V(G_1)$ to $V(G_2)$ such that $u_1v_1 \in E(G_1)$ if and only if $\phi(u_1)\phi(v_1) \in E(G_2)$. In such a case we write $G_1 \cong G_2$ and ϕ is called an *isomorphism* from G_1 to G_2 . An *automorphism* of a graph G is an isomorphism of the graph G onto itself. A graph G is *vertex transitive* if, for any two vertices u and v , there is an element g in the automorphism group of G such that $g(u) = v$.

A graph H is called a *subgraph* of a graph G , written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G .

Suppose that V' is a nonempty subset of V . The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of G . For a vertex v of a nontrivial graph G , the subgraph $G \setminus v$ consists of all vertices of G except v and all edges of G except those incident with v . For a proper subset U of $V(G)$, the subgraph $G \setminus U$ has vertex set $V(G) \setminus U$ and its edge set consists of all edges of G joining two vertices in $V(G) \setminus U$; $G \setminus U$ is an induced subgraph of G .

If the vertices of a graph G of order n can be labeled (or relabeled) v_1, v_2, \dots, v_n so that its edges are $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$, then G is called a *path*, and is denoted by P_n .

If the vertices of a graph G of order $n \geq 3$ can be labeled (or relabeled) v_1, v_2, \dots, v_n so that its edges are $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$, and v_1v_n , then G is called a *cycle*, and is denoted by C_n . Note that a *k-cycle* is a cycle of length k . Furthermore, the *girth* of G is the length of a shortest cycle in G ; if G has no cycles we define the girth of G to be infinite.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n . A graph G is *connected* if every two vertices of G are connected, that is, if G contains a $u - v$ path for every pair u, v of distinct vertices of G . A graph G that is not connected is called *disconnected*. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a *component* of G . A graph G is then connected if and only if it has exactly one component.

The *complement* \overline{G} of a graph G is a graph whose vertex set is $V(G)$ and such that for each pair u, v of vertices of G , uv is an edge of \overline{G} if and only if uv is not an edge of G . A *clique* of a graph G is a subset W of V such that $G[W]$ is complete. Clearly, W is a clique of G if and only if W is an independent set of \overline{G} , and so the two concepts are complementary.

A subset M of E is called a *matching* in G if no two are incident in G ; the two ends of an edge in M are said to be *matched under* M . A matching M *saturates* a vertex v , and v is said to be *M-saturated*, if some edge of M is incident with v ; otherwise, v is *M-unsaturated*. If every vertex of G is M -saturated, the matching M is *perfect*.

A *bipartite graph* is a graph whose vertex set can be partitioned into two subsets X and Y , so that each edge has one end in X and one end in Y ; such a partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of

Y ; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

A graph G is called a *planar graph* if G can be drawn in the plane so that no two of its edges cross each other. A graph G is *regular of degree r* if $\deg(v) = r$ for each vertex v of G . Such graphs are called *r -regular*. A 3-regular graph is also called a *cubic graph*.

The *Cartesian product* of graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$, in which $(g, h) \sim (g', h')$ if either both $g = g'$ and $h \sim h'$ or both $h = h'$ and $g \sim g'$.

A *polynomial algorithm* for graphs is one whose execution time is bounded by a polynomial in either the number of edges or the number of vertices. An *NP-complete problem* is a problem having a ‘yes’ or ‘no’ answer that can be solved nondeterministically in polynomial time, and all other such problems can be transformed to it in polynomial time. Such problems are generally accepted as being computationally difficult.

Note that all of these terms are from Bondy and Murty [18], Gross and Yellen [21] and Chartrand and Zhang [15]; more terms will be defined later as required.

1.2 Overview of the Thesis

In this thesis, we determine when G is well-covered for $G = C(n, S)$ in one of the following classes.

Class 1: $S = \{1, 2, \dots, d\}$ where $1 \leq d \leq \frac{n}{2}$.

Class 2: $S = \{d + 1, d + 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ where $1 \leq d \leq \frac{n-2}{2}$.

Class 3: $S = \{1, 2, \dots, d\} \cup \{\lfloor \frac{n}{2} \rfloor\}$ where $1 \leq d \leq \frac{n}{2}$.

Class 4: $S = \{2, 4, \dots, 2d\}$ where $1 \leq d \leq \frac{n}{4}$.

Class 5: $S = \{1, 3, 5, \dots, 2d + 1\}$ where $0 \leq d \leq \frac{n-2}{4}$.

Class 6: $S = \{1\} \cup \{2, 4, \dots, 2d\}$ where $1 \leq d \leq \frac{n}{4}$.

Class 7: $S = \{2, 3, \dots, d\}$ where $2 \leq d \leq \frac{n}{2}$.

Class 8: $S = \{1\} \cup \{3, 4, \dots, d\}$ where $3 \leq d \leq \frac{n}{2}$.

Class 9: $S = \{1\} \cup \{4, 5, \dots, d\}$ where $4 \leq d \leq \frac{n}{2}$.

Class 10: $S = \{3, 4, \dots, d\}$ where $3 \leq d \leq \frac{n}{2}$.

Class 11: $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$ where $A \subseteq S$ such that $|A| = 1$.

Class 12: $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$ where $A \subseteq S$ such that $|A| = 2$.

Class 13: $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$ where $A \subseteq S$ such that $|A| = 3$.

In Chapter 2, we provide a brief survey of well-covered graphs and of well-covered circulant graphs. We also provide an overview of the various findings regarding the closure of well-covered (circulant) graphs under graph products.

In Chapter 3, we first investigate Classes 1 and 2. Necessary and sufficient conditions for members of these families to be well-covered were determined by Hoshino [33, 34] using the independence polynomial. Note that the *independence polynomial* $I(G, x)$ is $\sum_{k=0}^n i_k x^k$, where i_k is the number of independent sets of cardinality k in G [33]. We will provide proofs for Hoshino's two classes using an alternative approach. This approach allows us to generalize his results by allowing us to determine exactly which of these graphs is 1-well-covered. Finally, we characterize Class 3 and we also determine which of these graphs is 1-well-covered.

In Chapter 4, we investigate Classes 4, 5, and 6, and in Chapter 5, we investigate Classes 7, 8, 9, and 10. Necessary and sufficient conditions for members of these families to be well-covered are determined using various approaches.

In Chapter 6, we investigate Classes 11, 12, and 13. By examining the maximal cliques in the complementary graphs we are able to characterize these classes.

In Chapter 7, we provide concluding remarks and several open questions pertaining to well-covered circulant graphs.

Chapter 2

Preliminary Results

In this chapter, we provide a brief survey of well-covered graphs. In relation to the notion of well-coveredness, employing a girth approach has been proven to be a very successful technique, and consequently significant findings regarding this concept are accredited to its usage. In view of the computational complexity issues regarding well-covered graphs, work in this area has concentrated on characterizing various subclasses of well-covered graphs. Well-covered cubic graphs, 1-well-covered graphs, claw-free graphs and very well-covered graphs are among those subclasses that are reviewed in this chapter.

We also present an overview of the findings regarding well-covered circulant graphs. Though considerable research has been conducted on well-covered graphs, very little has been done on determining whether a circulant graph is well-covered. Hoshino's [33, 34] findings are primarily the only literature found regarding well-covered circulant graphs. Necessary and sufficient conditions for certain families of circulant graphs to be well-covered are outlined in this section.

The Cartesian product, the categorical product, the lexicographic product and the strong product of graphs are among the products that are surveyed. Many techniques are used to produce new graphs from existing ones and using graph products is a well known approach. An infinite family of well-covered (circulant) graphs can be generated by applying the lexicographic product and those families can also be extended by applying the Cartesian product.

2.1 Well-Covered Graphs

Given a graph G , one can pose the following question:

Is G well-covered, and if not, can we easily show that it is not well-covered?

The complexity of this recognition problem has been independently investigated by Chvátal and Slater [43] and by Sankaranarayanan and Stewart [35]. They independently proved the following result.

Theorem 2.1 *The following decision problem is NP-complete: “Is a given graph G not a well-covered graph?”*

In addition, Caro, Sebö and Tarsi [44] proved the following result.

Theorem 2.2 *It is co-NP-complete to determine whether a given graph G is well-covered. This remains true when restricted to graphs with no $K_{1,4}$ induced subgraph.*

Though different restrictions were employed by Caro [45], similar results of co-NP-completeness were established by him. Note that a graph G is said to be Z_m -well-covered, if the cardinality of every maximal independent set of vertices is congruent to the same number modulo m [46].

Theorem 2.3 *The decision problem “is G well-covered?” remains co-NP-complete even if it is given that G is both $K_{1,3m+1}$ -free and Z_m -well-covered for any fixed positive integer $m \geq 1$.*

The following propositions are fundamental results in the study of well-covered graphs.

Proposition 2.4 *A graph G is well-covered if and only if every component of G is well-covered.*

Proof. Clear. ■

Proposition 2.5 [39] *Let G be a graph. Suppose that I is an independent set of G . If G is well-covered, then every component of $G \setminus N[I]$ is well-covered.*

Proof. We prove the contrapositive of the above statement; that is, if any component of $G \setminus N[I]$ is not well-covered, then G is not well-covered.

Suppose that H_1 is a component of $G \setminus N[I]$ and that H_1 is not well-covered. Then there exists two maximal independent sets, in H_1 , of differing cardinality, say I_1 and I'_1 . Let the remaining components of $G \setminus N[I]$ be H_j , if any, where $j = 2, \dots, k$. Choose a maximal independent set I_j for H_j for each of $j = 2, \dots, k$. Then G has two independent sets $I \cup I_1 \cup I_2 \cup \dots \cup I_k$ and $I \cup I'_1 \cup I_2 \cup \dots \cup I_k$ of differing cardinalities. Hence, G is not well-covered. ■

Since circulant graphs are vertex transitive the above result implies:

Corollary 2.6 *Let G be a circulant graph on n vertices and v be a vertex in G . Then G is well-covered if and only if $H = G \setminus N[v]$ is well-covered. Furthermore, $\beta(G) = \beta(H) + 1$.*

2.2 1-Well-Covered Graphs

A well-covered graph G is 1-well-covered if $G - v$ is well-covered for all $v \in V(G)$ [19]. These graphs were introduced by Staples [19] in her 1975 dissertation. The graphs in Figure 2.1 are a few examples of 1-well-covered graphs. Note that the first two graphs, $C(5, \{1\})$ and $C(7, \{1, 3\})$, are 1-well-covered circulant graphs.

Staples [19, 20] defined W_n -graphs. For a positive integer n , a graph G belongs to class W_n if any n disjoint independent sets in G can be expanded to n disjoint maximum independent sets. She showed that 1-well-covered graphs and W_2 -graphs are identical.

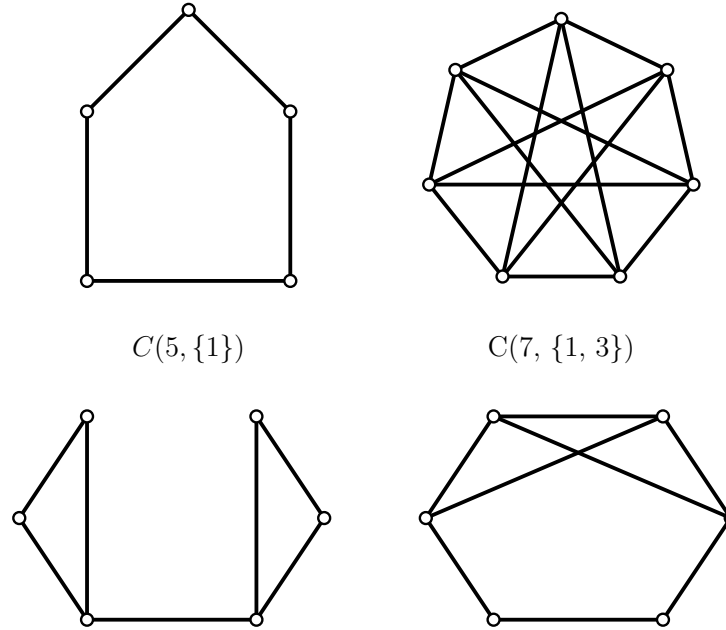


Figure 2.1: 1-well-covered graphs [20].

Later, Pinter [25] examined W_2 -graphs in his 1991 dissertation. He characterized cubic 1-well-covered graphs [26], 3-connected 4-regular planar 1-well-covered graphs [26], and planar 1-well-covered graphs with girth four [27]. Pinter [28] also considered 1-well-covered graphs with girth four and he introduced two types of constructions which enabled him to produce an infinite family of 1-well-covered graphs with girth four. Hartnell [9] characterized 1-well-covered graphs with no 4-cycles. Recently Finbow and Hartnell [5] characterized triangle free 1-well-covered graphs and they introduced various constructions which enabled them to produce an infinite family of 1-well-covered graphs.

Caro, Sebö and Tarsi [44] proved the following result.

Theorem 2.7 *For any positive integer n , recognizing a member of W_n is co-NP-complete.*

In addition, Caro [45] proved the following result.

Theorem 2.8 [45] *For any positive integer n , recognizing a member of W_n is co-NP-complete even for $K_{1,4}$ -free graphs.*

2.3 Well-Covered Cubic Graphs

Well-covered cubic graphs of connectivity at most 2 were characterized by Campbell [39] in his 1987 dissertation. Campbell and Plummer [40] characterized cubic, planar, 3-connected, well-covered graphs. They established that there are precisely four graphs satisfying this property and they are shown in Figure 2.2. Later, Campbell, Ellingham and Royle [41] constructed an infinite family of well-covered cubic graphs, and they established that there exists six other graphs shown in Figure 2.3. In view of the findings of Finbow, Hartnell and Nowakowski [2], they showed that the graph P_{14} , shown in Figure 2.3, is the only cubic graph of girth at least five. They also considered cubic graphs of girth three and four. Subsequently, they established a characterization of well-covered cubic graphs.

In addition, Campbell, Ellingham and Royle [41] established the following result.

Theorem 2.9 *The problem of recognizing well-covered cubic graphs is solvable in polynomial time.*

2.4 Very Well-Covered Graphs

A graph G is said to be *very well-covered* if every maximal independent set has cardinality $|V|/2$ [37]. These graphs have been independently characterized by Nelson and Staples in 1973 (see [19]) and by Favaron [31] in 1982.

For simplicity we use Plummer's [30] definition of *property P*.

Definition 2.4.1 *Suppose a graph G has a perfect matching F . Then the matching $F = \{a_1b_1, \dots, a_nb_n\}$ has property P if*

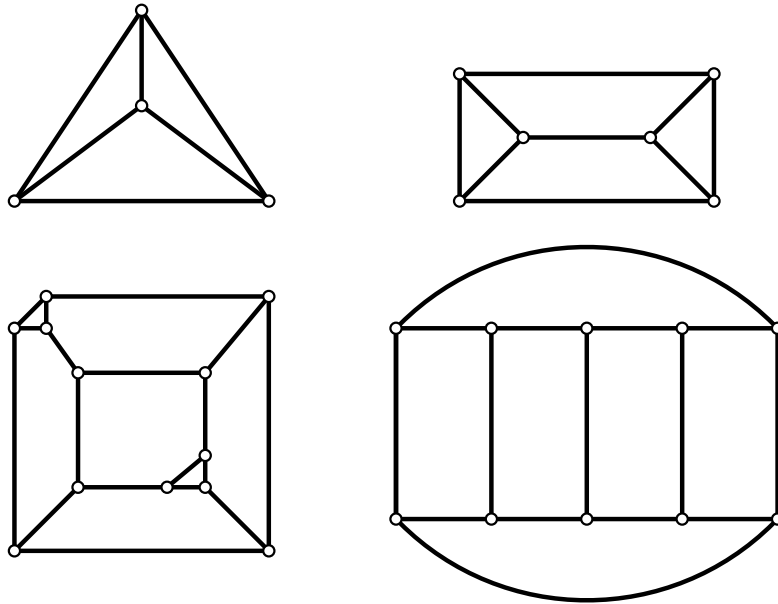


Figure 2.2: Cubic, planar, 3-connected, well-covered graphs [40].

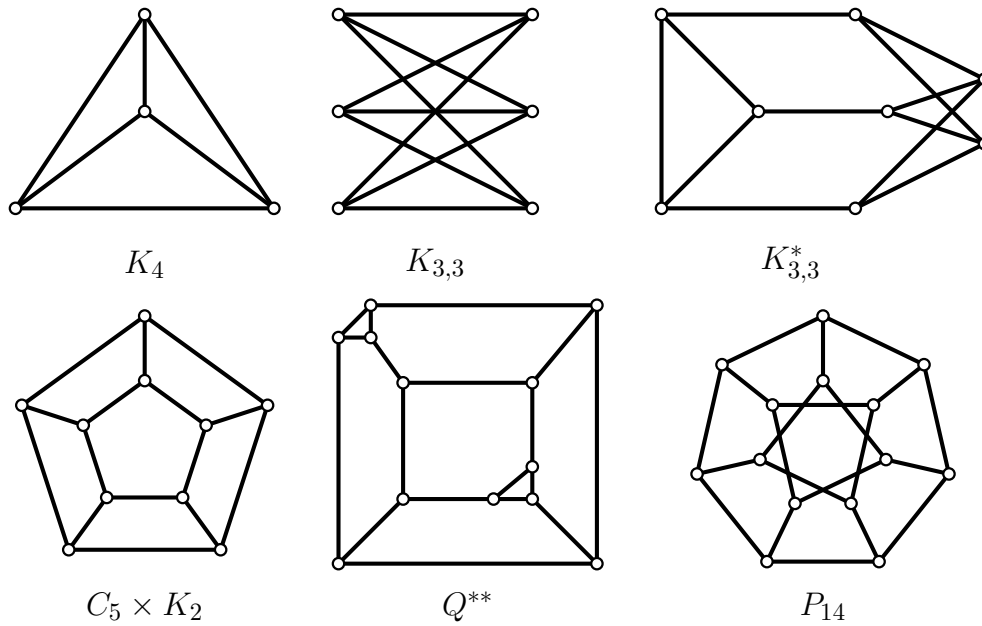


Figure 2.3: Well-covered cubic graphs [41].

- (i) no point $\omega \in G$ satisfies $\omega \sim a_i$ and $\omega \sim b_i$ where $a_i b_i \in F$; and
- (ii) no set of two independent points $\{u, v\} \subseteq V(G)$ satisfies $u \sim a_i$ and $v \sim b_i$ where $a_i b_i \in F$.

The characterization of Favaron is as follows.

Theorem 2.10 [31] *For a graph G , the following properties are equivalent:*

- (a) G is very well-covered.
- (b) There exists a perfect matching in G which satisfies the property P .
- (c) There exists at least one perfect matching in G , and every perfect matching of G satisfies P .

Note that Staples [19] established parts (a) and (b) of Theorem 2.10.

Well-covered bipartite graphs, in particular well-covered trees, were independently characterized by Staples [19] and Ravindra [16]. Later, Sankaranarayana and Stewart [36, 37] characterized very well-covered graphs.

The following theorem is a fundamental result in the study of very well-covered graphs.

Theorem 2.11 *A graph G is very well-covered if and only if every component of G is very well-covered.*

2.5 Claw-Free Graphs

A graph G is said to be a claw-free graph if G contains no induced subgraph isomorphic to $K_{1,3}$ [10]. Whitehead [10] characterized well-covered claw-free graphs containing no four cycles, and King [14] characterized 3-connected claw-free planar well-covered graphs.

In addition, Hartnell and Plummer [6] and King [42] investigated 4-regular, 4-connected, claw-free, well-covered graphs. The classes \mathcal{G}_0 , \mathcal{G}_1 , \mathcal{G}_2 and $H_{r,n}$ are introduced.

Definition 2.5.1 A **Harary graph** $H_{r,n}$ is defined for integers r and n with $2 \leq r < n$ such that $H_{r,n}$ has order n . Let $V(H_{r,n}) = \{v_1, v_2, \dots, v_n\}$, then $H_{r,n}$ is constructed as follows:

(i) r is even.

Let $r = 2k$. For each integer i where $1 \leq i \leq n$, the vertex v_i is adjacent to $v_{i+1}, v_{i+2}, \dots, v_{i+k}$ and to $v_{i-1}, v_{i-2}, \dots, v_{i-k}$. Thus, $H_{r,n}$ is an r -regular graph of order n .

(ii) r is odd and n is even.

Let $r = 2k + 1$ and $n = 2l$. For each integer i where $1 \leq i \leq n$, the vertex v_i is joined to the $2k$ vertices mentioned above as well as to v_{i+l} . Thus, $H_{r,n}$ is an r -regular graph of order n .

(iii) r and n are both odd.

Let $r = 2k + 1$ and $n = 2l + 1$. In this case $H_{r,n}$ is obtained from $H_{r-1,n}$ by adding the edge $v_i v_{i+l}$ for each i where $l + 2 \leq i \leq n + 1$. Therefore, when r and n are both odd, $H_{r,n}$ contains one vertex of degree $r + 1$ and $n - 1$ vertices of degree r .

If r or n is even, $H_{r,n}$ is a circulant graph and is of size $\frac{rn}{2}$; while if r and n are odd, then the size of $H_{r,n}$ is $\frac{rn+1}{2}$. In general, the size of $H_{r,n}$ is $\lceil \frac{rn}{2} \rceil$.

- The class \mathcal{G}_0 consists of the graph K_5 and all graphs constructed as follows. Let $K_4(1), \dots, K_4(r)$ be any collection of at least two vertex-disjoint K_4 s. Now

join these K_4 s with a perfect matching to obtain a graph on $4r$ vertices which is 4-connected, 4-regular and claw-free [6].

- The class \mathcal{G}_1 consists of the set of 4-regular Harary graphs, $H_{4,k}$, where $k \geq 6$ [42].
- The class \mathcal{G}_2 can be described as the class of 4-regular 4-connected claw-free graphs in which each vertex lies on two edge-disjoint triangles [6].

Hartnell and Plummer [6] established the following results.

Lemma 2.12 *Suppose $G \in \mathcal{G}_0$ and $G \neq K_5$. Then G is well-covered if and only if each K_4 in G is joined by edges to no more than three other K_4 s.*

Lemma 2.13 *Let $G \in \mathcal{G}_2$. Then G is well-covered if and only if $G = L(K_{3,3})$.*

King [42] later established the following result.

Lemma 2.14 *The graphs $H_{4,5}, H_{4,6}, H_{4,7}, H_{4,8}$, and $H_{4,11}$, (shown in Figure 2.4), are the only 4-regular, well-covered Harary Graphs.*

The following theorem is a summary of Hartnell, Plummer and King's findings.

Theorem 2.15 [6, 42] *There are infinitely many 4-regular, 4-connected, claw-free, well-covered graphs in the class \mathcal{G}_0 , five in the class \mathcal{G}_1 , and one in the class \mathcal{G}_2 .*

In addition, Hartnell and Plummer [6] proved the following result.

Theorem 2.16 *There are precisely five 4-connected claw-free planar well-covered graphs. They are the two graphs shown in Figure 2.5, the graph shown in Figure 2.6, and the Harary graphs $H_{4,6}$ and $H_{4,8}$ shown in Figure 2.4.*

Tankus and Tarsi [12, 13] established the following result.

Theorem 2.17 *Well-covered claw-free graphs can be recognized in polynomial time.*

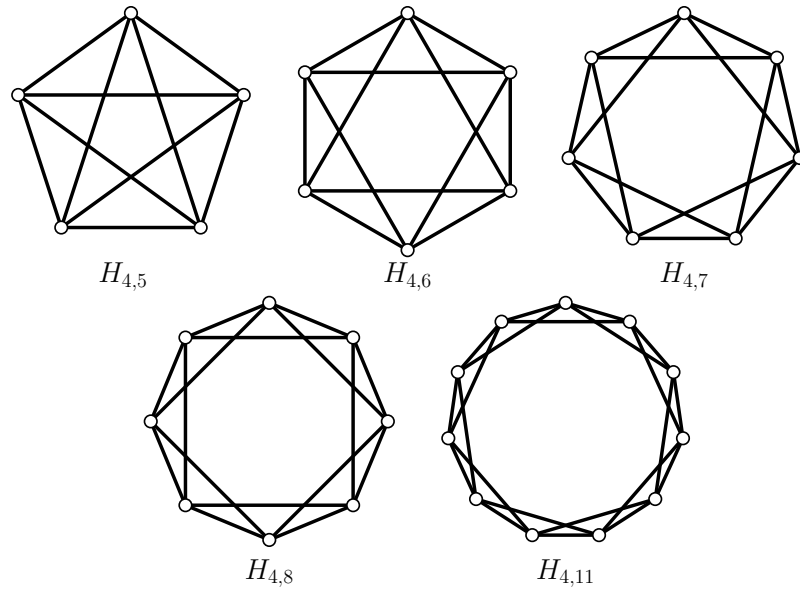


Figure 2.4: The well-covered, 4-regular Harary graphs [42].

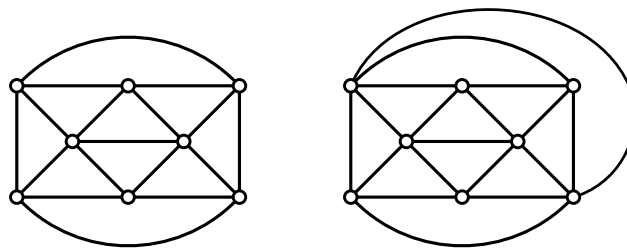


Figure 2.5: 4-connected claw-free planar well-covered graphs [6].

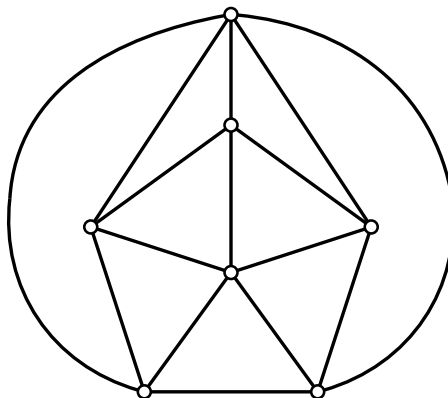


Figure 2.6: A 4-connected claw-free planar well-covered graph from [6].

2.6 Well-Covered Property: A Girth Approach

The study of the well-coveredness property versus girth was initially introduced by Finbow and Hartnell [1] in the course of their investigation of a 2-person game. The aim of this inquiry was to determine a winning strategy for any graph regardless of how the players moved. As a result, a characterization of well-covered graphs with girth at least eight was established. Finbow, Hartnell and Nowakowski characterized well-covered graphs of girth at least five [2], and well-covered graphs containing neither 4- nor 5- cycles [3]. Later, Finbow and Hartnell [4] investigated “Parity graphs”; those that do not contain cycles of order five or less were characterized. Gasquoine, Hartnell, Nowakowski and Whitehead [38] introduced techniques for constructing well-covered graphs with no 4-cycles. Hartnell [7] examined the local structure of well-covered graphs without 4-cycles. Recently Brown, Nowakowski and Zverovich [22] examined the structure of well-covered graphs with no cycles of length four. Caro and Hartnell [46] characterized Z_m -well-covered graphs of girth at least six and weighted well-covered graphs of girth at least seven.

2.7 Well-Covered Circulant Graphs

A circulant graph is said to be well-covered if every maximal independent set is a maximum independent set. The problem of determining whether a circulant graph is well-covered was originally investigated by R. Hoshino [33] in his 2007 dissertation. The isomorphism problem for circulant graphs has been investigated by various scholars for over three decades. Li [11] provided an excellent survey regarding the isomorphism of circulant graphs. A complete solution of the isomorphism problem for circulants of arbitrary order was recently established by Muzychuk [24].

An example of all circulant graphs on seven nodes is illustrated in Figure 2.7.

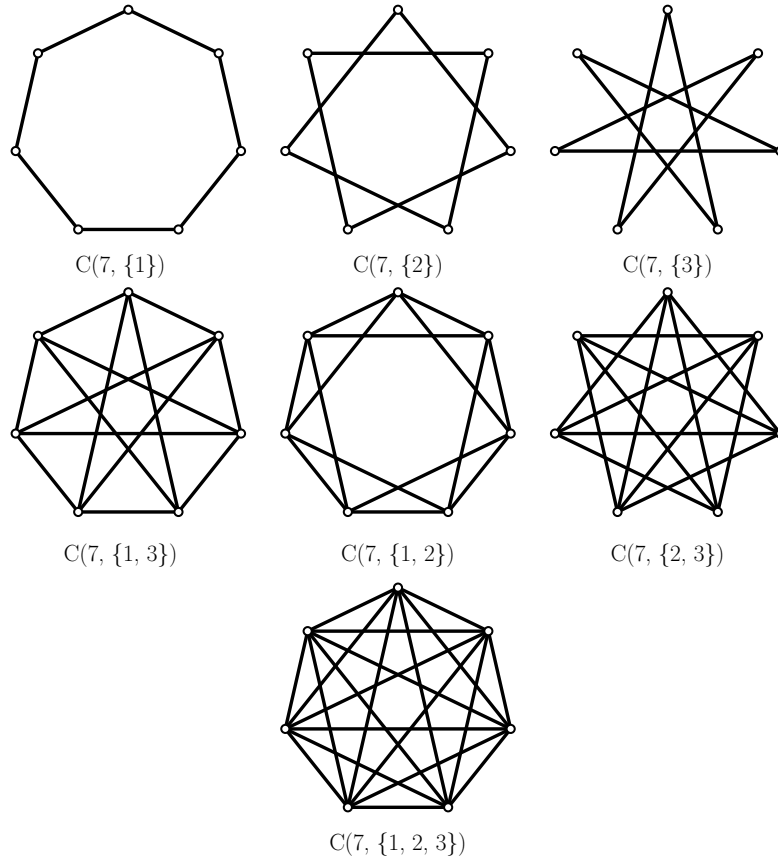


Figure 2.7: All Circulant Graphs on seven nodes.

Note that $C(7, \{1\}) \cong C(7, \{2\}) \cong C(7, \{3\})$ and $C(7, \{1, 2\}) \cong C(7, \{1, 3\}) \cong C(7, \{2, 3\})$. In addition, $C(7, \{1, 2, 3\})$ is the complete graph K_7 .

Definition 2.7.1 If $S \subseteq \mathbb{Z}_n$, we define $-S = \{y \in \mathbb{Z}_n : y + s \equiv 0 \text{ for some } s \in S\}$ and set $\langle S \rangle = S \cup (-S)$. We say that $T, S \subseteq \mathbb{Z}_n$ are equivalent if $\langle T \rangle = \langle S \rangle$. If r is an integer then $rs = \{rs \in \mathbb{Z}_n \mid s \in S\}$.

Hoshino and Brown [34] investigated the class of circulant graphs on n vertices with the following generating set: $S = \{1, 2, \dots, d\}$ and $S = \{d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor\}$. They established necessary and sufficient conditions for members of these families to be well-covered using the “independence polynomial” (defined on page 6).

Theorem 2.18 *Let n and d be integers with $n \geq 2d$ and $d \geq 1$. Then $G = C_{n, \{1, 2, \dots, d\}}$ is well-covered if and only if $n \leq 3d + 2$ or $n = 4d + 3$.*

Theorem 2.19 *Let n and d be integers with $n \geq 2d + 2$ and $d \geq 1$. Then $G = C_{n, \{d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor\}}$ is well-covered if and only if $n > 3d$ or $n = 2d + 2$.*

In Chapter 3, we will provide proofs for their two theorems using an alternative approach. This approach allows us to generalize their results by allowing us to determine exactly which of these graphs is 1-well-covered.

Definition 2.7.2 *Let $k \geq 1$ and let (a_1, a_2, \dots, a_k) be a k -tuple of integers with each $a_i \geq 3$. Define $n_0 = 1$, and $n_i = a_i n_{i-1} - 1$, for $1 \leq i \leq k$. Then for each $1 \leq j \leq i \leq k$, set*

$$S_{j,i} = \begin{cases} \pm S_{j,i-1} \pmod{n_{i-1}} & \text{for all } 1 \leq j < i \\ \{1, 2, \dots, \lfloor \frac{n_i}{2} \rfloor\} - \bigcup_{j=1}^{i-1} S_{j,i} & \text{for } j = i \end{cases}$$

Then $G_j(a_1, a_2, \dots, a_k)$ is defined to be $C(n_k, S_{j,k})$, the circulant graph on n_k vertices with generating set $S_{j,k}$ [33].

Note that $G_{j,k}$ is just an abbreviation of $G_j(a_1, a_2, \dots, a_k)$.

Hoshino characterized the well-covered graphs in this family as follows.

Theorem 2.20 *Let (a_1, a_2, \dots, a_k) be a k -tuple of positive integers, with each $a_i \geq 3$. Define $G_{j,k}$ for each k -tuple. Then $G_{j,k}$ is well-covered if and only if $j = k$, or $(j, k) = (1, 2)$ with $a_2 \leq 4$.*

In view of the findings of Finbow, Hartnell and Nowakowski [2, 3], Hoshino [33] established the following results.

Theorem 2.21 *Let G be a connected well-covered circulant graph of girth $g \geq 5$. Then G is isomorphic to K_1 , K_2 , C_5 , or C_7 .*

Theorem 2.22 *Let G be a connected well-covered circulant graph containing neither C_4 or C_5 as a subgraph. Then G is isomorphic to K_1 , K_2 , K_3 , or C_7 .*

Based on the characterization established by Campbell, Ellingham and Royle [41], Hoshino and Brown [34] determined all connected well-covered cubic graphs that are circulants.

Theorem 2.23 *Let G be a connected circulant cubic graph. Then G is well-covered if and only if it is isomorphic to one of the following graphs: $C(4, \{1, 2\})$, $C(6, \{1, 3\})$, $C(6, \{2, 3\})$, $C(8, \{1, 4\})$, or $C(10, \{2, 5\})$.*

Theorem 2.24 *Let $G = C(2n, \{a, n\})$, where $1 \leq a < n$. Let $t = \gcd(2n, a)$. Then G is well-covered if and only if $\frac{2n}{t} \in \{3, 4, 5, 6, 8\}$.*

In addition, Hoshino and Brown [34] proved the following result.

Theorem 2.25 *Let $G = C(n, S)$ be an arbitrary circulant graph. Then it is co-NP-complete to determine whether G is well-covered.*

2.8 Graph Products

In addition to the Cartesian product (defined on page 5), the following three products will be useful for our discussion. In the *categorical product graph* $G \times H$, the vertices (g, h) and (g', h') are adjacent if $g \sim g'$ and $h \sim h'$. In the *lexicographic product graph* $G[H]$, the vertices (g, h) and (g', h') are adjacent if either $g \sim g'$ or both $g = g'$ and $h \sim h'$. In the *strong product graph* $G \otimes H$, the vertices (g, h) and (g', h') are adjacent either if both $g \sim g'$ and $h = h'$, if both $g = g'$ and $h \sim h'$ or if both $g \sim g'$ and $h \sim h'$ [33].

The closure of circulant graphs under graph products has been independently investigated by Broere and Hattingh [17] and by Hoshino [33]. According to Hoshino

the closure of circulant graphs is not preserved under the categorical product or the strong product.

With respect to the Cartesian product of cycles, Broere and Hattingh [17] established the following result.

Theorem 2.26 $C_p \square C_q$ is a circulant graph if and only if p and q are relatively prime.

More generally Hoshino [33] attained the following result.

Theorem 2.27 Let $G = C(n, S_1)$ and $H = C(m, S_2)$ be circulant graphs. Define S to be the set of integers k in $\{1, 2, \dots, \lfloor \frac{nm}{2} \rfloor\}$ that satisfy one of the following conditions:

(i) $k = im$, for some $i \in S_1$.

(ii) $k = jn$, for some $j \in S_2$.

Then $G \square H$ is isomorphic to the circulant $C(nm, S)$.

With respect to the lexicographic product, Broere and Hattingh [17] established the following result.

Theorem 2.28 Let G and H be circulants of order p and q respectively. Then $G[H]$ is a circulant of order pq .

Similar results were also attained by Hoshino [33].

Theorem 2.29 Let $G = C(n, S_1)$ and $H = C(m, S_2)$ be circulant graphs. Define

$$S = \left(\bigcup_{t=0}^{\lfloor \frac{m-1}{2} \rfloor} tn + S_1 \right) \cup \left(\bigcup_{t=1}^{\lfloor \frac{m}{2} \rfloor} tn - S_1 \right) \cup nS_2$$

where $tn \pm S_1 = \{tn \pm r : r \in S_1\}$ and $nS_2 = \{nq : q \in S_2\}$.

Then $G[H]$ is isomorphic to the circulant $C(nm, S)$.

As a corollary to Theorem 2.29, Hoshino proved the following result.

Corollary 2.30 *For each n and m , the graph $C_n[\overline{K_m}]$ is a circulant.*

The preservation of the well-coveredness property under graph products is of interest amongst researchers. This notion has been investigated by Topp and Volkmann [23] and by Fradkin [32]. The categorical product, also known as the conjunction of graphs, is one of the graph products that was examined by Topp and Volkmann. Though necessary conditions for the categorical product of two graphs to be (very) well-covered were established in [23], the results won't be stated here in view of the fact that circulant graphs are not closed under this product.

Next, despite the fact that various special cases were examined by Topp and Volkmann [23], a complete characterization of when the Cartesian product of two graphs is well-covered was not attained. They established the following results.

Theorem 2.31 *If G_1, G_2 are connected bipartite graphs and each of them is different from K_1 , then $G_1 \square G_2$ is well-covered if and only if $G_1 = G_2 = K_2$.*

Theorem 2.32 *If G_1, G_2 are connected very well-covered graphs, then $G_1 \square G_2$ is very well-covered if and only if $G_1 = G_2 = K_2$.*

Theorem 2.33 *For all positive integers n and m , $K_n \square K_m$ is well-covered.*

Theorem 2.34 *The Cartesian product $C_n \square C_m$ of cycles C_n and C_m is well-covered if and only if $n = 3$ or $m = 3$.*

Furthermore, the question of when the Cartesian product of two graphs is well-covered was recently investigated by Fradkin [32]. Primarily, she explored the necessary conditions under which that property is not preserved and proved the following results.

Theorem 2.35 *If G is a non-well-covered graph, then $G \square G$ is not well-covered.*

Theorem 2.36 *Let G be any graph of girth ≥ 5 . Then $G \square G$ is not well-covered.*

With respect to the lexicographic product the following result proven by Topp and Volkmann [23] is significant primarily due to the fact that circulant graphs are closed under this product.

Theorem 2.37 *Let G be a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ a family of nonempty graphs indexed by the vertices of G . Then the lexicographic product $G[\mathcal{H}]$ is a well-covered graph if and only if G and \mathcal{H} satisfy the following two conditions:*

- (1) *each graph H_v of the family \mathcal{H} is well-covered,*
- (2) *$\sum_{v \in S_G} \alpha(H_v) = \sum_{u \in S'_G} \alpha(H_u)$ for every two maximal independent sets S_G and S'_G of G .*

As a corollary to Theorem 2.37, Topp and Volkmann deduced the following result.

Corollary 2.38 *The lexicographic product $G[H]$ of two nonempty graphs G and H is a well-covered graph if and only if G and H are well-covered graphs; if graphs G and H are nonempty and one of them is without isolated vertices, then the lexicographic product $G[H]$ is very well-covered if and only if exactly one of G and H is very well-covered and the second is totally disconnected, i.e., without edges.*

Regarding the notion of well-covered circulants and graph products one can pose the following question.

Given that two graphs, G and H , are well-covered circulants, is it possible to find a product under which the resulting new graph is also a well-covered circulant graph?

From Theorem 2.29 and Corollary 2.38, we can deduce that the lexicographic product of two well-covered circulants is also a well-covered circulant graph. Furthermore, from Theorem 2.26 and Theorem 2.34 we can deduce that the Cartesian product of cycles C_n and C_m is a well-covered circulant whenever n and m are relatively prime with $n = 3$ or $m = 3$. Consequently, an infinite family can be generated by applying these products and this result is significant to extending our own families of well-covered circulants.

Chapter 3

Characterization of Well-Covered Graphs in Classes 1, 2, and 3

In this chapter, we investigate the class of circulant graphs on n vertices with a generating set S , where S is one of $\{1, 2, \dots, d\}$, $\{d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor\}$, or $\{1, 2, 4, \dots, 2d\}$. These classes were originally investigated by Hoshino and Brown [34]. They established necessary and sufficient conditions for members of these families to be well-covered using the independence polynomial. We will provide proofs for their two classes using an alternative approach. This approach allows us to generalize their results by allowing us to determine exactly which of these graphs is 1-well-covered. Finally, we characterize the class of circulant graphs on n vertices with a generating set $S = \{1, 2, \dots, d, \lfloor \frac{n}{2} \rfloor\}$; we also determine which of these graphs is 1-well-covered.

Ravindra, in 1977, examined well-covered graphs and his results are as follows.

Theorem 3.1 [16] *Let G be a connected bipartite graph. Then G is well-covered if and only if G contains a perfect matching F such that for every edge uv in F , $N(u) \cup N(v)$ is a complete bipartite graph.*

Corollary 3.2 *Let G be a bipartite graph that contains a perfect matching F such that for every edge uv in F , $N(u) \cup N(v)$ is a complete bipartite graph. Then G is well-covered.*

Proof. By Theorem 3.1, each component of G is well-covered. Hence, by Theorem 2.11, G is well-covered. ■

A characterization of the well-covered graphs in Class 1 can now be stated.

Theorem 3.3 *Let n and d be integers with $1 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{1, 2, \dots, d\})$ is well-covered if and only if one of the following conditions holds:*

(i) $n \leq 3d + 2$, or

(ii) $n = 4d + 3$,

and G is 1-well-covered if and only if $n = 2d$, $n = 2d + 1$ or $2d + 3 \leq n \leq 3d + 2$.

Furthermore, if $n = 2d$ or $n = 2d + 1$ then $\beta(G) = 1$; if $2d + 2 \leq n \leq 3d + 2$ then $\beta(G) = 2$; and if $n = 4d + 3$ then $\beta(G) = 3$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. Let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$.

(i) $n \leq 3d + 2$.

If $n = 2d$ or $n = 2d + 1$ then $H = \emptyset$, G is well-covered and $\beta(G) = 1$. Clearly G is also 1-well-covered.

We now consider the case where $2d + 2 \leq n \leq 3d + 2$. We claim that H is a complete graph. Since $N_G(v_0) = \{v_1, v_2, \dots, v_d\} \cup \{v_{-1}, v_{-2}, \dots, v_{-d}\}$, $v_i \in V(H)$ implies that $d + 1 \leq i \leq 2d + 1$. To begin, if $|V(H)| = 1$ then H is a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$ and G is well-covered. Next, we consider the case where $2 \leq |V(H)| \leq d + 1$. Let $v_i, v_j \in V(H)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then,

$$d + 1 \leq i < j \leq 2d + 1$$

$$0 \leq j - i \leq (2d + 1) - (d + 1)$$

$$0 \leq j - i \leq d$$

Since $j - i \in S$, H is a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$ by Corollary 2.6. Furthermore, since $|V(H)| > 1$, G is 1-well-covered.

(ii) $n = 4d + 3$.

Let $H_1 = G[\{v_i: i = d + 1, \dots, 2d, 2d + 1\}]$ and $H_2 = G[\{v_i: i = -(2d + 1), -2d, \dots, -(d + 1)\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

We claim that H_i is a complete graph for $i = 1$ and 2 . Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$\begin{aligned} d + 1 &\leq i < j \leq 2d + 1 \\ 0 &\leq j - i \leq (2d + 1) - (d + 1) \\ 0 &\leq j - i \leq d \end{aligned}$$

Hence, v_i and v_j are adjacent, and thus H_1 is a complete graph. Similarly, H_2 is a complete graph.

Note that $V(H) \neq \emptyset$, hence $|I \cap V(H)| \neq \emptyset$. Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(d+1)}$, a vertex in $V(H_2)$. We claim that $|k - (-(d + 1))| > d$. We have,

$$\begin{aligned} d + 1 &\leq k \leq 2d + 1 \\ (d + 1) + (d + 1) &\leq k - (-(d + 1)) \leq (2d + 1) + (d + 1) \\ 2d + 2 &\leq k - (-(d + 1)) \leq 3d + 2 \end{aligned}$$

Since $n - (2d + 2) = 2d + 1 = \lfloor \frac{n}{2} \rfloor$ and $n - (3d + 2) = d + 1$, we can deduce

that $v_{-(d+1)} \not\sim v_k$. Hence, $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Therefore, $|I| = 3$, G is well-covered and $\beta(G) = 3$.

We finally note that $H \setminus \{v_{-(d+1)}\} \subseteq N[v_{2d+2}]$. Hence, G is not 1-well-covered.

We now proceed to prove the ‘only if’ direction.

Case 3.3.1 $3d + 2 < n < 4d + 3$.

Let $I_1 = \{v_0, v_{\lfloor \frac{n}{2} \rfloor}\}$. Consider any vertex $v_i \in V(H)$ such that $d < i < \lfloor \frac{n}{2} \rfloor$. We claim that $v_{\lfloor \frac{n}{2} \rfloor}$ is adjacent to v_i . We have

$$\begin{aligned} -\lfloor \frac{n}{2} \rfloor &< -i < -d \\ 0 &< \lfloor \frac{n}{2} \rfloor - i < \lfloor \frac{n}{2} \rfloor - d \end{aligned}$$

Also, given our assumption that $3d + 2 < n < 4d + 3$, we have the following:

$$\begin{aligned} \lfloor \frac{3d+2}{2} \rfloor &\leq \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{4d+3}{2} \rfloor \\ \lfloor \frac{3d+2}{2} \rfloor - d &\leq \lfloor \frac{n}{2} \rfloor - d \leq \lfloor \frac{4d+3}{2} \rfloor - d \\ \lfloor \frac{d}{2} \rfloor + 1 &\leq \lfloor \frac{n}{2} \rfloor - d \leq d + 1 \end{aligned}$$

Hence, $\lfloor \frac{n}{2} \rfloor - i \in S$. Similarly, if we consider any vertex $v_i \in V(H)$ such that $\lfloor \frac{n}{2} \rfloor + 1 < i < n - (d + 1)$, we can show that $v_{\lfloor \frac{n}{2} \rfloor} \sim v_i$. If n is even, then $\lfloor \frac{n}{2} \rfloor \equiv -\lfloor \frac{n}{2} \rfloor$; and if n is odd, then $\lfloor \frac{n}{2} \rfloor + 1 \equiv -\lfloor \frac{n}{2} \rfloor$. We also note that $v_{\lfloor \frac{n}{2} \rfloor} \not\sim v_0$ since $|\lfloor \frac{n}{2} \rfloor - 0| > d + 1$. Therefore, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_{d+1}, v_{-(d+1)}\}$. Since $|(d+1) - 0| = d+1 \notin \langle S \rangle$ and $|0 - (-(d+1))| = d+1 \notin \langle S \rangle$, we conclude that v_0 is adjacent to neither v_{d+1} nor $v_{-(d+1)}$. We now show that $v_{d+1} \not\sim v_{-(d+1)}$. Note that $|(d+1) - (-(d+1))| = 2d+2$. We have

$$3d + 2 < n < 4d + 3$$

$$(3d + 2) - (2d + 2) < n - (2d + 2) < (4d + 3) - (2d + 2)$$

$$d < n - (2d + 2) < 2d + 1$$

Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 3.3.2 $n > 4d + 3$.

Let $I' = \{v_{2d+1}, v_{-(d+2)}\}$. We verify that $v_{2d+1} \not\sim v_{-(d+2)}$. Note that $|(2d + 1) - (-(d + 2))| = 3d + 3$. Given our assumption that $n > 4d + 3$, it follows that $n - (3d + 3) > d$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_i : i = -1, 0, \dots, d\}$. Let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_d\}$. Notice that $v_d \not\sim v_{-1}$ since $|d - (-1)| = d + 1 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. ■

A characterization of the well-covered graphs in Class 2 can now be stated.

Theorem 3.4 *Let n and d be integers with $1 \leq d \leq \frac{n-2}{2}$. Then $G = C(n, \{d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor\})$ is well-covered if and only if $n > 3d$ or $n = 2d+2$, and G is 1-well-covered if and only if $n > 3$ where $d = 1$ or $n = 2d + 2$. Furthermore, $\beta(G) = d + 1$.*

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. Let I be a maximal independent set of G .

Case 3.4.1 $n = 2d + 2$.

It follows that $G = C(2d + 2, \{d + 1\})$. Let $v_i, v_j \in V(G)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Since v_i and v_j are connected in G if and only if $|j - i| = d + 1$, G is therefore a collection of edge-disjoint K_2 s. Hence, G is well-covered and $\beta(G) = d + 1$. Clearly G is also 1-well-covered.

Case 3.4.2 $n > 3d$.

Without loss of generality, assume that v_0 is in I and let H be the graph induced by $G \setminus N[v_0]$. Let $H_1 = G[\{v_i : i = 1, 2, \dots, d\}]$ and $H_2 = G[\{v_i : i = -d, \dots, -2, -1\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then $1 \leq i < j \leq d$ and $0 \leq j - i \leq d - 1$. Since $j - i \notin \langle S \rangle$, we conclude that $V(H_1)$ is an independent set. Similarly, $V(H_2)$ is an independent set. By symmetry, we can also deduce that $|V(H_1)| = |V(H_2)|$, and thus H is a bipartite graph.

Claim 3.4.1 We claim that H contains a perfect matching F such that for every edge $e = uv$ in F , $N(u) \cup N(v)$ is a complete bipartite graph. Let $F = \{v_{1+j}v_{-d+j} \mid 0 \leq j \leq d - 1\}$. Observe that $v_{1+j} \sim v_{-d+j}$ since $|(1 + j) - (-d + j)| = d + 1 \in S$.

For each edge j , $0 \leq j \leq d - 1$, we note that

$$N_H(v_{1+j}) = \{v_{-d+k} \mid 0 \leq k < 1 + j\}; \text{ and}$$

$$N_H(v_{-d+j}) = \{v_s \mid j < s \leq d\}.$$

To complete the proof of the claim we show that if $0 \leq k < 1 + j$ and $j < s \leq d$, then $|s - (-d + k)| \in \langle S \rangle$.

Observe that we have $j + d < s + d \leq 2d$ and $j + d - k < s + d - k \leq 2d - k$. But $k \leq j$, thus $j + d - k \geq d$ and $d + 1 \leq s + d - k \leq 2d - k$, completing the proof of Claim 3.4.1.

Therefore, by Corollary 3.2, H is well-covered and $\beta(H) = d$, and thence by Corollary 2.6, G is well-covered and $\beta(G) = d + 1$.

Observe that when $d = 1$, H is isomorphic to K_2 and hence G is 1-well-covered. We then may assume that $d > 1$.

Note that $H \setminus \{v_{-1}\}$ is a bipartite graph containing an odd number of vertices (greater than one) and thence by Theorem 3.1, G is not 1-well-covered. This establishes the ‘if’ direction.

We now proceed to prove the ‘only if’ direction.

For $d = 1$ and 2 , $2d + 2 \geq 3d$, hence we may assume that $d \geq 3$. Let $I' = \{v_d, v_{-d}\}$. We verify that $v_d \not\sim v_{-d}$. Note that $|d - (-d)| = 2d$. Given our assumption that $2d + 2 < n \leq 3d$, it follows that $2 < n - 2d \leq d$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . Observe that

$$V(H_1) = v_0 \cup \left\{ v_i : d + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\} \cup \left\{ v_i : -\left\lfloor \frac{n}{2} \right\rfloor \leq i \leq -(d + 1) \right\}.$$

Let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 .

Next, let $K_2 = \{v_{-(d+1)}, v_{d+1}\}$. We verify that $v_{d+1} \not\sim v_{-(d+1)}$. Note that $|(d + 1) - (-(d + 1))| = 2d + 2$. We have

$$2d + 2 < n \leq 3d$$

$$(2d + 2) - (2d + 2) < n - (2d + 2) \leq (3d) - (2d + 2)$$

$$0 < n - (2d + 2) \leq d - 2$$

Hence, K_2 is an independent set in G with cardinality greater than that of K_1 , and thus G is not well-covered. ■

Note that Theorems 3.3 and the following theorem provide us with a characterization of the well-covered, circulant, Harary graphs (see Definition 2.5.1).

Theorem 3.5 *Let n and d be integers with $1 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{1, 2, \dots, d\} \cup \{\lfloor \frac{n}{2} \rfloor\})$ is well-covered if and only if one of the following conditions holds:*

(i) $n \leq 3d + 2$, or

(ii) $d \geq 2$ and $4d + 1 \leq n \leq 4d + 5$, or

(iii) G is one of the following: $C(6, \{1, 3\})$, $C(7, \{1, 3\})$, $C(8, \{1, 4\})$ or $C(11, \{1, 5\})$.

Furthermore, if $2d \leq n \leq 2d + 3$ then $\beta(G) = 1$; if $2d + 4 \leq n \leq 3d + 2$ or $G = C(7, \{1, 3\})$ then $\beta(G) = 2$; and if $4d + 1 \leq n \leq 4d + 5$ and $d \geq 2$, or if G is one of $C(6, \{1, 3\})$, $C(8, \{1, 4\})$ or $C(11, \{1, 5\})$ then $\beta(G) = 3$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. Let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Let $H_1 = G[\{v_i : i = d+1, d+2, \dots, \lfloor \frac{n}{2} \rfloor - 1\}]$ and $H_2 = G[\{v_i : i = -(\lfloor \frac{n}{2} \rfloor - 1), \dots, -(d+2), -(d+1)\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

(i) $n \leq 3d + 2$.

If $2d \leq n \leq 2d + 3$ then $H = \emptyset$, G is well-covered and $\beta(G) = 1$. Clearly G is also 1-well-covered.

We now consider the case where $2d + 4 \leq n \leq 3d + 2$. We claim that H is a complete graph. Since $N_G(v_0) = \{v_1, v_2, \dots, v_d, v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_{-1}, v_{-2}, \dots, v_{-d}, v_{-\lfloor \frac{n}{2} \rfloor}\}$, $v_i \in V(H)$ implies that $i \geq d + 1$. Observe that $2 \leq |V(H)| \leq d + 1$. Let v_i ,

$v_j \in V(H)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Since $n \leq 3d + 2$, we can deduce that $j \leq (3d + 2) - (d + 1) = 2d + 1$. Then,

$$\begin{aligned} d + 1 &\leq i < j \leq 2d + 1 \\ 0 &\leq j - i \leq (2d + 1) - (d + 1) \\ 0 &\leq j - i \leq d \end{aligned}$$

Since $j - i \in S$, H is a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$ by Corollary 2.6. Furthermore, since $|V(H)| > 1$, G is 1-well-covered.

(ii) $d \geq 2$ and $4d + 1 \leq n \leq 4d + 5$.

We claim that H_i is a complete graph for $i = 1$ and 2 . Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$\begin{aligned} d + 1 &\leq i < j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 0 &\leq j - i \leq \left\lfloor \frac{n}{2} \right\rfloor - d - 2 \end{aligned}$$

Since $4d + 1 \leq n \leq 4d + 5$, it follows that $d - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor - d - 2 \leq d$. Hence, v_i and v_j are adjacent, therefore H_1 is a complete graph. Similarly, H_2 is a complete graph. Note that $V(H) \neq \emptyset$, hence $|I \cap V(H)| \neq \emptyset$. Now we consider the following three cases.

Case 3.5.1 $n = 4d + 1$ or $n = 4d + 2$.

Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(d+1)}$, a vertex

in $V(H_2)$. We claim that $v_{-(d+1)} \not\sim v_k$. We have

$$\begin{aligned} d+1 &\leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ (d+1) + (d+1) &\leq k - (-(d+1)) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 + (d+1) \\ 2d+2 &\leq k - (-(d+1)) \leq \left\lfloor \frac{n}{2} \right\rfloor + d \end{aligned}$$

If $n = 4d + 1$ then $n - (\lfloor \frac{n}{2} \rfloor + d) = d + 1$, and thence $v_{-(d+1)} \not\sim v_k$. And if $n = 4d + 2$ then $n - (\lfloor \frac{n}{2} \rfloor + d) = d + 1$, and thence $v_{-(d+1)} \not\sim v_k$.

Therefore, $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Therefore, $|I| = 3$, G is well-covered and $\beta(G) = 3$.

We finally note that $H \setminus \{v_{-(d+1)}\} \subseteq N[v_{1-\lfloor \frac{n}{2} \rfloor}]$. Hence, G is not 1-well-covered.

Case 3.5.2 $n = 4d + 3$ or $n = 4d + 4$.

Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(d+2)}$ a vertex in $V(H_2)$. We claim that $v_{-(d+2)} \not\sim v_k$. We have

$$\begin{aligned} d+1 &\leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ (d+1) + (d+2) &\leq k - (-(d+2)) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 + (d+2) \\ 2d+3 &\leq k - (-(d+2)) \leq \left\lfloor \frac{n}{2} \right\rfloor + d+1 \end{aligned}$$

If $n = 4d + 3$ then $n - (\lfloor \frac{n}{2} \rfloor + d + 1) = d + 1$, and thence $v_{-(d+2)} \not\sim v_k$. And if $n = 4d + 4$ then $n - (\lfloor \frac{n}{2} \rfloor + d + 1) = d + 1$, and thence $v_{-(d+2)} \not\sim v_k$.

Therefore, $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Therefore, $|I| = 3$, G is well-covered and $\beta(G) = 3$.

We finally note that $H \setminus \{v_{-(d+2)}\} \subseteq N[v_{1-\lfloor \frac{n}{2} \rfloor}]$. Hence, G is not 1-well-covered.

Case 3.5.3 $n = 4d + 5$.

Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(d+3)}$ a vertex in $V(H_2)$. We claim that $v_{-(d+3)} \not\sim v_k$. We have,

$$\begin{aligned} d+1 &\leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ (d+1) + (d+3) &\leq k - (-(d+3)) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 + (d+3) \\ 2d+4 &\leq k - (-(d+3)) \leq \left\lfloor \frac{n}{2} \right\rfloor + d+2 \end{aligned}$$

Since $n = 4d + 5$, we can deduce that $\left\lfloor \frac{n}{2} \right\rfloor + d + 2 = 3d + 4$. Observe that $n - (3d + 4) = d + 1$, and thence $v_{-(d+3)} \not\sim v_k$. Hence, $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Therefore, $|I| = 3$, G is well-covered and $\beta(G) = 3$.

We finally note that $H \setminus \{v_{-(d+3)}\} \subseteq N[v_{1-\lfloor \frac{n}{2} \rfloor}]$. Hence, G is not 1-well-covered.

(iii) G is one of the following $C(6, \{1, 3\})$, $C(7, \{1, 3\})$, $C(8, \{1, 4\})$ or $C(11, \{1, 5\})$.

Note that $C(7, \{1, 3\})$ is well-covered and $\beta(G) = 2$; and $C(6, \{1, 3\})$, $C(8, \{1, 4\})$ and $C(11, \{1, 5\})$ are well-covered and $\beta(G) = 3$.

To establish the ‘only if’ direction, first observe that $C(10, \{1, 5\})$ is not well-covered. Next, we consider the remaining cases.

Case 3.5.4 $d \geq 3$ and $3d + 2 < n < 4d + 1$.

Let $I_1 = \{v_0, v_{\lfloor \frac{n}{2} \rfloor - 1}\}$. Consider any vertex $v_i \in V(H)$ such that $d + 1 \leq i <$

$\lfloor \frac{n}{2} \rfloor - 1$. We claim that $v_{\lfloor \frac{n}{2} \rfloor - 1}$ is adjacent to v_i . We have

$$\begin{aligned} 1 - \lfloor \frac{n}{2} \rfloor &< -i \leq -(d+1) \\ 0 &< \lfloor \frac{n}{2} \rfloor - 1 - i \leq \lfloor \frac{n}{2} \rfloor - d - 2 \end{aligned}$$

Also, given our assumption that $3d + 2 < n < 4d + 1$, we have the following:

$$\begin{aligned} \lfloor \frac{3d+2}{2} \rfloor &\leq \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{4d+1}{2} \rfloor \\ \lfloor \frac{3d+2}{2} \rfloor - d - 2 &\leq \lfloor \frac{n}{2} \rfloor - d - 2 \leq \lfloor \frac{4d+1}{2} \rfloor - d - 2 \\ \lfloor \frac{d}{2} \rfloor - 1 &\leq \lfloor \frac{n}{2} \rfloor - d - 2 \leq d - 1 \end{aligned}$$

Hence, $v_{\lfloor \frac{n}{2} \rfloor - 1} \sim v_i$. Similarly, if we consider any vertex $v_i \in V(H)$ such that $\lfloor \frac{n}{2} \rfloor < i \leq \lfloor \frac{n}{2} \rfloor - 1 + d$, we can show that $v_{\lfloor \frac{n}{2} \rfloor - 1} \sim v_i$. We also note that $v_{\lfloor \frac{n}{2} \rfloor - 1} \not\sim v_0$ since $|(\lfloor \frac{n}{2} \rfloor - 1) - 0| > d$. Therefore, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_{-d}, v_1, v_{\lfloor \frac{n}{2} \rfloor - 1}\}$. Observe that v_1 is adjacent to neither v_{-d} nor $v_{\lfloor \frac{n}{2} \rfloor - 1}$ since $|1 - (-d)| = d+1 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor - 1) - 1| = \lfloor \frac{n}{2} \rfloor - 2 \notin \langle S \rangle$. Furthermore, v_{-d} is not adjacent to $v_{\lfloor \frac{n}{2} \rfloor - 1}$ since $|(\lfloor \frac{n}{2} \rfloor - 1) - (-d)| = \lfloor \frac{n}{2} \rfloor + d - 1 \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 3.5.5 $d \geq 2$ and $4d + 6 \leq n \leq 5d + 4$.

Let $I' = \{v_{-(2d+1)}, v_{d+2}\}$. First, we show that $v_{-(2d+1)} \not\sim v_{d+2}$. Note that $|(d+2) - (-(2d+1))| = 3d+3$. Given our assumption that $4d+6 \leq n \leq 5d+4$, it follows that $d+3 \leq n - (3d+3) \leq 2d+1$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-(d-1)}, \dots, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent

set in H_1 . Next, let $K_2 = \{v_{-d}, v_1\}$. Observe that $v_{-d} \not\sim v_1$ since $|1 - (-d)| = d + 1 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 3.5.6 $d \geq 2$ and $5d + 5 \leq n \leq 6d + 3$.

Let $I' = \{v_{-(2d+1)}, v_{2d+1}\}$. First, we show that $v_{-(2d+1)} \not\sim v_{2d+1}$. Note that $|(2d + 1) - (-(2d + 1))| = 4d + 2$. Given our assumption that $5d + 5 \leq n \leq 6d + 3$, it follows that $d + 3 \leq n - (4d + 2) \leq 2d + 1$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-(d-1)}, \dots, v_0, v_1, \dots, v_{d-1}, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_d\}$. Observe that $v_{-d} \not\sim v_d$ since $|d - (-d)| = 2d \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 3.5.7 $d \geq 2$ and $6d + 4 \leq n \leq 6d + 7$.

We first note that $C(10, \{1, 5\})$, $C(12, \{1, 6\})$, and $C(13, \{1, 5\})$ are not well-covered. Furthermore, one can verify that the eight graphs that arise when $d = 2$ and 3 are not well-covered.

We now consider the case where $d \geq 4$. Let $I' = \{v_{-(2d+1)}, v_{2d}, v_{3d+1}\}$. Observe that $v_{3d+1} \not\sim v_{2d}$ since $|(3d + 1) - 2d| = d + 1 \notin \langle S \rangle$. Next, we show that $v_{-(2d+1)}$ is adjacent to neither v_{2d} nor v_{3d+1} . Note that $|(2d) - (-(2d + 1))| = 4d + 1$ and $|(3d + 1) - (-(2d + 1))| = 5d + 2$. Given our assumption that $6d + 4 \leq n \leq 6d + 7$, it follows that $2d + 3 \leq n - (4d + 1) \leq 2d + 6$ and $d + 2 \leq n - (5d + 2) \leq d + 5$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-(d-1)}, \dots, v_0, v_1, \dots, v_{d-1}\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal

independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_{d-1}\}$. Observe that $v_{-d} \not\sim v_{d-1}$ since $|(d-1) - (-d)| = 2d-1 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 3.5.8 n is even and $n \geq 6d+8$.

Let $J = \{v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 2}, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + d + 2}\}$. Observe that $v_{\lfloor \frac{n}{2} \rfloor}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 2}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 2}$ since $|\lfloor \frac{n}{2} \rfloor - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + d + 1 \notin \langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - (d+1)| = \lfloor \frac{n}{2} \rfloor - d - 1 \notin \langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - (\lfloor \frac{n}{2} \rfloor - d - 2)| = d + 2 \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + d + 2 - \lfloor \frac{n}{2} \rfloor| = d + 2 \notin \langle S \rangle$. Next, we note that $v_{\lfloor \frac{n}{2} \rfloor - d - 2}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 2}$ since $|\lfloor \frac{n}{2} \rfloor - d - 2 - (-(d+1))| = \lfloor \frac{n}{2} \rfloor - 1 \notin \langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - d - 2 - (d+1)| = \lfloor \frac{n}{2} \rfloor - 2d - 3 \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + d + 2 - (\lfloor \frac{n}{2} \rfloor - d - 2)| = 2d + 4 \notin \langle S \rangle$. We also note that $v_{\lfloor \frac{n}{2} \rfloor + d + 2}$ is adjacent to neither $v_{-(d+1)}$ nor v_{d+1} since $|\lfloor \frac{n}{2} \rfloor + d + 2 - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + 2d + 3 \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + d + 2 - (d+1)| = \lfloor \frac{n}{2} \rfloor + 1 \notin \langle S \rangle$. Furthermore, $v_{-(d+1)}$ is not adjacent to v_{d+1} since $|(d+1) - (-(d+1))| = 2d + 2 \notin \langle S \rangle$. Hence, J is an independent set in G . We extend J to a maximal independent set I_1 in G .

For $d = 1$, we let $I_2 = \{I_1 \setminus v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_0, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}\}$. Observe that v_0 is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 2}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 2}$. Next, we note that $v_{\lfloor \frac{n}{2} \rfloor - d - 2}$ is adjacent to neither $v_{\lfloor \frac{n}{2} \rfloor - 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|\lfloor \frac{n}{2} \rfloor - 1 - (\lfloor \frac{n}{2} \rfloor - d - 2)| = d + 1 \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + 1 - (\lfloor \frac{n}{2} \rfloor - d - 2)| = d + 3 \notin \langle S \rangle$. We also note that $v_{\lfloor \frac{n}{2} \rfloor + d + 2}$ is adjacent to neither $v_{\lfloor \frac{n}{2} \rfloor - 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|\lfloor \frac{n}{2} \rfloor + d + 2 - (\lfloor \frac{n}{2} \rfloor - 1)| = d + 3 \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + d + 2 - (\lfloor \frac{n}{2} \rfloor + 1)| = d + 1 \notin \langle S \rangle$. Furthermore, $v_{\lfloor \frac{n}{2} \rfloor - 1}$ is adjacent to neither $v_{d+1}, v_{-(d+1)}$ nor $v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|\lfloor \frac{n}{2} \rfloor - 1 - (d+1)| = \lfloor \frac{n}{2} \rfloor - d - 2 \notin \langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - 1 - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + d \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + 1 - (\lfloor \frac{n}{2} \rfloor - 1)| = 2 \notin S$; and $v_{\lfloor \frac{n}{2} \rfloor + 1}$ is adjacent to neither v_{d+1} nor $v_{-(d+1)}$ since $|\lfloor \frac{n}{2} \rfloor + 1 - (d+1)| = \lfloor \frac{n}{2} \rfloor - d \notin \langle S \rangle$ and $|\lfloor \frac{n}{2} \rfloor + 1 - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + d + 2 \notin \langle S \rangle$. Hence, I_2 is an independent set in

G with cardinality greater than that of I_1 , and thus G is not well-covered.

On the other hand for $d \geq 2$, we let $I_3 = \{I_1 \setminus v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_0, v_{\lfloor \frac{n}{2} \rfloor - 1}\}$. Hence, I_3 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 3.5.9 n is odd and $n \geq 6d + 9$.

Case 3.5.9.1 $d = 1$.

Let $J = \{v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + d + 3}\}$. Note that $v_{\lfloor \frac{n}{2} \rfloor - d - 3}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ since $|(\lfloor \frac{n}{2} \rfloor - d - 3) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor - 2 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor - d - 3) - (d+1)| = \lfloor \frac{n}{2} \rfloor - 2d - 4 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor - 1) - (\lfloor \frac{n}{2} \rfloor - d - 3)| = d + 2 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + 1) - (\lfloor \frac{n}{2} \rfloor - d - 3)| = d + 4 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (\lfloor \frac{n}{2} \rfloor - d - 3)| = 2d + 6 \notin \langle S \rangle$. We also note that $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + 2d + 4 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (d+1)| = \lfloor \frac{n}{2} \rfloor + 2 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (\lfloor \frac{n}{2} \rfloor - 1)| = d + 4 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (\lfloor \frac{n}{2} \rfloor + 1)| = d + 2 \notin \langle S \rangle$. Hence, J is an independent set in G . We extend J to a maximal independent set I_1 in G .

Let $I_2 = \{I_1 \setminus v_{\lfloor \frac{n}{2} \rfloor + 1}\} \cup \{v_0, v_{\lfloor \frac{n}{2} \rfloor + 2}\}$. Observe that v_0 is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$. Next, we note that $v_{\lfloor \frac{n}{2} \rfloor + 2}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}, v_{\lfloor \frac{n}{2} \rfloor - 1}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ since $|(\lfloor \frac{n}{2} \rfloor + 2) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + d + 3 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + 2) - (d+1)| = \lfloor \frac{n}{2} \rfloor - d + 1 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + 2) - (\lfloor \frac{n}{2} \rfloor - d - 3)| = d + 5 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor + 2) - (\lfloor \frac{n}{2} \rfloor - 1)| = 3 \notin S$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (\lfloor \frac{n}{2} \rfloor + 2)| = d + 1 \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 3.5.9.2 $d = 2$.

Let $J = \{v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + d + 3}\}$. Observe that $v_{\lfloor \frac{n}{2} \rfloor}$ is adjacent to neither $v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}$ nor $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ since $|(\lfloor \frac{n}{2} \rfloor) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + d + 1 \notin$

$\langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - (d+1)| = \lfloor \frac{n}{2} \rfloor - d - 1 \notin \langle S \rangle$, $|\lfloor \frac{n}{2} \rfloor - (\lfloor \frac{n}{2} \rfloor - d - 3)| = d + 3 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - \lfloor \frac{n}{2} \rfloor| = d + 3 \notin \langle S \rangle$. Next, we note that $v_{\lfloor \frac{n}{2} \rfloor - d - 3}$ is adjacent to neither $v_{-(d+1)}$, v_{d+1} nor $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ since $|(\lfloor \frac{n}{2} \rfloor - d - 3) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor - 2 \notin \langle S \rangle$, $|(\lfloor \frac{n}{2} \rfloor - d - 3) - (d+1)| = \lfloor \frac{n}{2} \rfloor - 2d - 4 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (\lfloor \frac{n}{2} \rfloor - d - 3)| = 2d + 6 \notin \langle S \rangle$. We also note that $v_{\lfloor \frac{n}{2} \rfloor + d + 3}$ is adjacent to neither $v_{-(d+1)}$ nor v_{d+1} since $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (-(d+1))| = \lfloor \frac{n}{2} \rfloor + 2d + 4 \notin \langle S \rangle$ and $|(\lfloor \frac{n}{2} \rfloor + d + 3) - (d+1)| = \lfloor \frac{n}{2} \rfloor + 2 \notin \langle S \rangle$. Hence, J is an independent set in G . We extend J to a maximal independent set I_1 in G .

Let $I_2 = \{I_1 \setminus v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_0, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}\}$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 3.5.9.3 $d > 2$.

Let $J = \{v_{-(d+1)}, v_{d+1}, v_{\lfloor \frac{n}{2} \rfloor - d - 3}, v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + d + 3}\}$. Hence, J is an independent set in G . We extend J to a maximal independent set I_1 in G .

Let $I_2 = \{I_1 \setminus v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_0, v_{\lfloor \frac{n}{2} \rfloor - 1}\}$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered. ■

Hartnell and Plummer [6] and King [42] examined 4-regular, 4-connected, claw-free graphs. Observe that the five Harary graphs stated in Theorem 2.15 are circulant graphs. The following corollary is a summary of Hartnell, Plummer and King's findings.

Corollary 3.6 [6, 42] *The graphs $C(5, \{1, 2\})$, $C(6, \{1, 2\})$, $C(7, \{1, 2\})$, $C(8, \{1, 2\})$, and $C(11, \{1, 2\})$ (shown in Figure 3.1) are the only 4-regular, 4-connected, claw-free well-covered circulant graphs .*

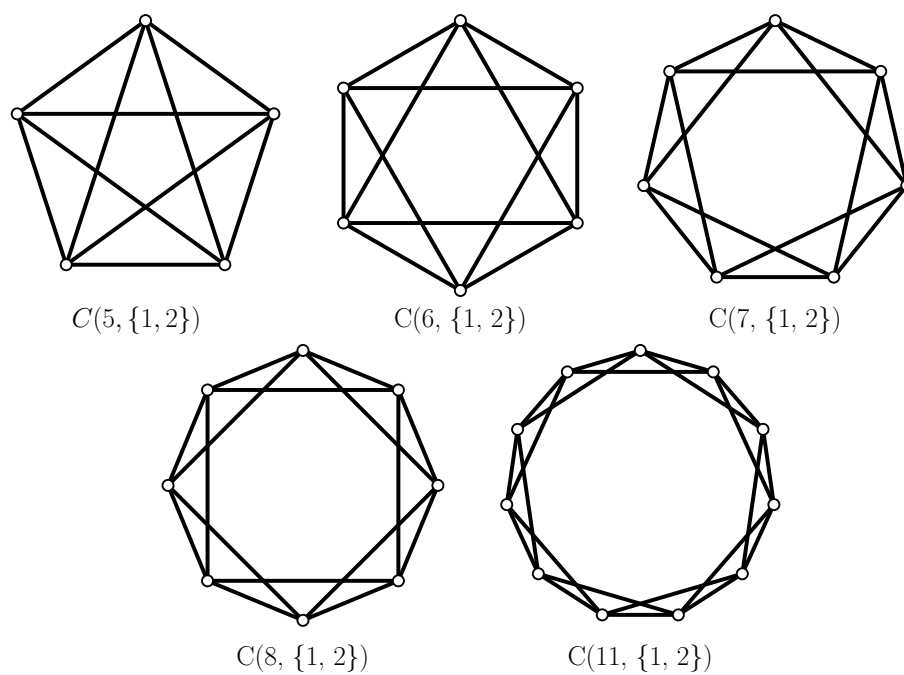


Figure 3.1: The well-covered, 4-regular Harary graphs [42].

Chapter 4

Characterization of Well-Covered Graphs in Classes 4, 5, and 6

In this chapter, we investigate the class of circulant graphs on n vertices with a generating set S , where S is one of $\{2, 4, \dots, 2d\}$, $\{1, 3, \dots, 2d+1\}$, or $\{1, 2, 4, \dots, 2d\}$. Necessary and sufficient conditions for members of these families to be well-covered are determined using various approaches.

Lemma 4.1 *Let $G = C(n, S)$ be a circulant graph where $n = rk$ and where the generating set $S = rP$. Then G is well-covered (very well-covered) if and only if $H = C(k, P)$ is well-covered (very well-covered). Furthermore, $\beta(G) = r\beta(H)$.*

Proof. Let $V(G) = \{v_i: i = 0, 1, \dots, rk - 1\}$, $V(H) = \{w_i: i = 1, \dots, k\}$ and the generating set of G be $S = rP$ where $P = \{a_1, a_2, \dots, a_q\}$ and $q \leq \lfloor \frac{n}{2} \rfloor$. We claim that G can be partitioned into r components G_t for $0 \leq t \leq r - 1$, where G_t is induced in G by

$$V(G_t) = [\{v_i \in G: i \equiv t \pmod{r}\}].$$

Suppose that $i \neq j$. Let $v_a \in G_i$ and $v_b \in G_j$. Since $a \not\equiv b \pmod{r}$, $v_a \not\sim v_b$ and hence for $0 \leq t < s \leq r - 1$, there are no edges between G_t and G_s .

Next, we show that each G_i is isomorphic to H . Define $\varphi: H \rightarrow G_t$ by setting $\varphi(w_i) = v_{ri+t}$ for all $w_i \in H$. If $\varphi(w_a) = \varphi(w_b)$, then $v_{a'} = v_{b'}$, where $a' = ra + t$ and $b' = rb + t$. So $ra + t \equiv rb + t \pmod{n}$ and hence $a \equiv b \pmod{n}$. Therefore, φ is one-to-one.

Now, let $v_w \in G_t$ and choose $s = \frac{w-t}{r}$. Then w_s is in H . Note that $\varphi(w_s) = v_{rs+t} = v_r\left(\frac{w-r}{t}\right)_{+t} = v_w$. So, φ is onto G_t .

We also need to verify that each edge $w_a w_b \in E(H)$ corresponds to an edge $\varphi(w_a)\varphi(w_b) \in E(G_t)$. Note that $\varphi(w_a) = v_{a'}$ and $\varphi(w_b) = v_{b'}$, where $a' = ra + t$ and $b' = rb + t$ so $a' - b' = r(a - b) \in r\langle p \rangle = \langle S \rangle$. Therefore, φ is an isomorphism. Hence, by Theorem 2.11, G is well-covered (very well-covered) if and only if $C(k, P)$ is well-covered (very well-covered). ■

A characterization of the well-covered graphs in Class 4 can now be stated.

Theorem 4.2 *Let n and d be integers with $1 \leq d \leq \frac{n}{4}$. Then $G = C(n, \{2, 4, \dots, 2d\})$ is well-covered if and only if one of the following conditions holds:*

(i) n is even and $4d \leq n \leq 6d + 4$, or

(ii) $n = 8d + 6$, or

(iii) $n = 4d + 3$, or

(iv) $G = C(5, \{2\})$.

Furthermore, if (i) holds $\beta(G) = 4$; if (ii) holds $\beta(G) = 6$; if (iii) holds $\beta(G) = 3$; and if (iv) holds $\beta(G) = 2$.

Proof. If n is even then by Lemma 4.1, G is well-covered if and only if $C\left(\frac{n}{2}, \{1, 2, \dots, d\}\right)$ is well-covered. Hence, in this case the theorem follows from Theorem 3.3.

We now proceed to prove the ‘if’ direction when n is odd.

First, we note that $C(5, \{2\})$ is well-covered and $\beta(G) = 2$.

Next, suppose that (iii) is true. Let $V(G) = \{v_i : i = 0, 1, \dots, n-1\}$ and let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Let $H_1 = G[\{v_i : i = 1, 3, \dots, 2d+1\}]$ and

$H_2 = G[\{v_i : i = -(2d+1), \dots, -3, -1\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

We claim that H_i is a complete graph for $i = 1$ and 2 . Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$\begin{aligned} 1 \leq i < j \leq 2d+1 \\ 0 \leq j-i \leq (2d+1) - 1 \\ 0 \leq j-i \leq 2d \end{aligned}$$

Observe that $j-i \leq 2d < \lfloor \frac{n}{2} \rfloor = 2d+1$. Since i and j are both odd, it follows that their difference is even. Hence, $j-i \in S$ and H_1 is a complete graph. Similarly, H_2 is a complete graph. Therefore, $\beta(H) \leq 2$.

Note that $V(H) \neq \emptyset$, hence $|I \cap V(H)| \neq \emptyset$. Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(2d+1)}$, a vertex in $V(H_2)$. We claim that $v_{-(2d+1)} \not\sim v_k$ for any $v_k \in V(H_1)$. Since $-(2d+1) \equiv (2d+2) \pmod{4d+3}$, it suffices to show that $v_{2d+2} \not\sim v_k$. We have

$$\begin{aligned} -(2d+1) \leq -k \leq -1 \\ (2d+2) - (2d+1) \leq (2d+2) - k \leq (2d+2) - 1 \\ 1 \leq (2d+2) - k \leq 2d+1 \end{aligned}$$

Since k is odd and $2d+2$ is even, we can deduce that their difference is odd and thence $v_{2d+2} \not\sim v_k$. Hence, $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Therefore, $|I| = 3$ and G is well-covered and $\beta(G) = 3$. This establishes the ‘if’ direction for n is odd.

To establish the ‘only if’ direction when n is odd, we consider the following two cases.

Case 4.2.1 $d > 1$ and $n = 4d + 1$.

Let $I_1 = \{v_0, v_1\}$. Note that $v_1 \not\sim v_0$ since $|1 - 0| = 1 \notin S$ and $|N[v_0] \cup N[v_1]| = 4d + 1$. Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_{2d-1}, v_{-(2d-1)}\}$. First, we note that $v_{2d-1} \not\sim v_0$ since $|(2d-1) - 0| = 2d-1 \notin \langle S \rangle$. Next, observe that $v_{2d-1} \not\sim v_{-(2d-1)}$ since $|(2d-1) - (-(2d-1))| = 4d-2 \equiv -3 \pmod{4d+1} \notin \langle S \rangle$. By symmetry, we can also deduce that $v_0 \not\sim v_{-(2d-1)}$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 4.2.2 $d \geq 1$ and $n \geq 4d + 5$.

Let $I' = \{v_0, v_1\}$. Note that $v_0 \not\sim v_1$ since $|1 - 0| = 1 \notin S$, hence I' is an independent set in G and $|N[v_0] \cup N[v_1]| = 4d + 2$.

Let $H_1 = G \setminus N_G[I']$. Given our assumption that $n \geq 4d + 5$, it follows that $n - (4d + 2) \geq 3$, hence $|V(H_1)| \geq 3$. Since n is odd and $(4d + 2)$ is even, it follows that their difference is odd, thus $|V(H_1)|$ is odd. Let $|V(H_1)| = 2k + 1$ where $k \geq 1$. Since $S = \{2, 4, \dots, 2d\}$, we let $K_1 = G[\{v_i : i = 2d + 2, 2d + 4, \dots, n - (2d + 1)\}]$ and $K_2 = G[\{v_i : i = 2d + 3, 2d + 5, \dots, n - (2d + 2)\}]$. Note that $V(K_1)$ together with $V(K_2)$ forms a partition of $V(H_1)$. Then $|V(K_1)| = k + 1$ and $|V(K_2)| = k$. Also note that

$$\begin{aligned} v_a \sim v_b &\Rightarrow a - b = 2c \text{ for some } c \\ &\Rightarrow a \equiv b \pmod{2}. \end{aligned}$$

Therefore, K_1 and K_2 are the two connected components of H_1 . If one of these components is not well-covered, then G is not well-covered. Hence, we assume that

both K_1 and K_2 are well-covered.

For $i = 1, 2$ let $G_i = C(2d+k+i, \{1, 2, \dots, d\})$ where $V(G_i) = \{v_0^i, v_1^i, \dots, v_{2d+k+i}^i\}$. Note that $K_i \cong G_i - N_{G_i}[v_0^i]$ for each i . Thus, K_i is well-covered if and only if G_i is well-covered. By Theorem 3.3, G_1 is well-covered if and only if $2d+k+1 \leq 3d+2$ or $2d+k+1 = 4d+3$; that is, if and only if $k \leq d+1$ or $k = 2d+2$. Similarly, G_2 is well-covered if and only if $2d+k+2 \leq 3d+2$ or $2d+k+2 = 4d+3$; that is, if and only if $k \leq d$ or $k = 2d+1$. Observe that if $k = 2d+2$, hence neither $k \leq d$ nor $k = 2d+1$ holds and therefore $k \leq d+1$. But then $k = 2d+1$ does not hold, and so for G_i to be well-covered we must have $k \leq d$. Therefore, $\beta(G_1) = \beta(G_2) = 2$, and hence $\beta(K_1) = \beta(K_2) = 1$. Choose vertices $u_i \in K_i$ for $i = 1, 2$. Note that $I' = \{v_0, v_1, u_1, u_2\}$ is a maximal independent set of size four in G . Hence,

$$\text{if } G \text{ is well-covered then } \beta(G) = 4 \quad (*)$$

Next, let $I'' = \{v_0, v_{2d+1}, v_{-(2d+1)}\}$. Observe that $v_{2d+1} \not\sim v_0$ since $|(2d+1) - 0| = 2d+1 \notin S$. By symmetry, we can also deduce that $v_0 \not\sim v_{-(2d+1)}$. Next, we show that $v_{2d+1} \not\sim v_{-(2d+1)}$. Note that $|(2d+1) - (-(2d+1))| = 4d+2$. Given our assumption that $n \geq 4d+5$, it follows that $n - (4d+2) \geq 3$. Since n is odd and $(4d+2)$ is even, it follows that their difference is odd, and hence, $v_{2d+1} \not\sim v_{-(2d+1)}$. Furthermore, v_{2d+1} is adjacent to all v_i 's such that $1 \leq i \leq 2d-1$ and $2d+3 \leq i \leq 4d+1$, where i is odd; and $v_{-(2d+1)}$ is adjacent to all v_i 's such that $-(4d+1) \leq i \leq -(2d+3)$ and $-(2d-1) \leq i \leq -1$, where i is even. Therefore, I'' is an independent set in G and $|N[v_0] \cup N[v_{2d+1}] \cup N[v_{-(2d+1)}]| \geq 4d+5$.

Let $H_2 = G \setminus N_G[I'']$. Given our assumption that $n \geq 4d+5$, it follows that $n - (4d+5) \geq 0$, hence $|V(H_2)| \geq 0$. Since n and $(4d+5)$ are both odd, we can deduce that their difference is even, hence $|V(H_2)|$ is even. Let $|V(H_2)| = 2p$ where

$p \geq 0$. Observe that for $p = 0$, I'' is a maximal independent set in G of size three which contradicts (*) if G is well-covered. Therefore we may assume that $p \geq 1$.

Now let $I''' = \{v_0, v_{2d+1}, v_{2d+2}, v_{-(2d+1)}, v_{-(2d+2)}\}$. Observe that since $p \geq 1$, we can deduce that $n \geq 6d + 5$. Note that v_{2d+2} is adjacent to neither v_0 nor v_{2d+1} since $|(2d+2)-0| = 2d+2 \notin \langle S \rangle$ and $|(2d+2)-(2d+1)| = 1 \notin S$. Next, we consider $v_{-(2d+2)}$. Note that $|(2d+1)-(-(2d+2))| = 4d+3$ and $|(2d+2)-(-(2d+2))| = 4d+4$. Given that $n \geq 6d+5$, it follows that $n-(4d+3) \geq 2d+2$ and $n-(4d+4) \geq 2d+1$. Hence, $v_{-(2d+2)}$ is adjacent to neither v_{2d+1} nor v_{2d+2} . By symmetry, we can also deduce that $v_{-(2d+2)}$ is adjacent to neither v_0 nor $v_{-(2d+1)}$ and $v_{2d+2} \not\sim v_{-(2d+1)}$. Furthermore, v_0 is adjacent to neither v_{2d+1} nor $v_{-(2d+1)}$ and $v_{2d+1} \not\sim v_{-(2d+1)}$. Therefore, I''' is an independent set in G of size five. Hence, in view of (*), G is not well-covered. ■

A characterization of the well-covered graphs in Class 5 can now be stated.

Theorem 4.3 *Let n and d be integers with $0 \leq d \leq \frac{n-2}{4}$. Then $G = C(n, \{1, 3, \dots, 2d+1\})$ is well-covered if and only if one of the following conditions holds:*

(i) *n is odd and $4d+3 \leq n \leq 6d+7$, or*

(ii) *n is even and either $n = 4d+2$ or $n = 4d+4$.*

Furthermore, if (i) holds $\beta(G) = \frac{n-2d-1}{2}$; if (ii) holds $\beta(G) = \frac{n}{2}$ and therefore G is very well-covered (see page 12).

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n-1\}$. First, we prove the ‘if’ direction. Let I be a maximal independent set of G . Without loss of generality, assume that v_0 is in I and let H be the graph induced by $G \setminus N[v_0]$.

(i) n is odd and $4d + 3 \leq n \leq 6d + 7$.

Set $n = 1 + 2(w + 2d)$ for $1 \leq w \leq d + 3$. Let $H_1 = G[\{v_i : i = 2, 4, \dots, 2(d + w - 1)\}]$ and $H_2 = G[\{v_i : i = -2(d + w - 1), \dots, -4, -2\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

We claim that $V(H_1)$ forms an independent set. Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then,

$$\begin{aligned} 2 \leq i < j &\leq 2(d + w - 1) \\ 0 \leq j - i &\leq 2(d + w - 1) - 2 \\ 0 \leq j - i &\leq 2d + 2w - 4 \end{aligned}$$

Observe that $j - i \leq 2d + 2w - 4 < n = 1 + 2(w + 2d)$. Note that for $w \leq 4$, $j - i \leq 2d + 2w - 4 \leq \lfloor \frac{n}{2} \rfloor = 2d + w$. On the other hand when $w \geq 5$, given our assumption that $n = 1 + 2(w + 2d)$, it follows that $n - (2d + 2w - 4) = 2d + 5 \leq \lfloor \frac{n}{2} \rfloor = 2d + w$. Since i and j are both even, it follows that their difference is even, hence $j - i \notin \langle S \rangle$. Therefore, $V(H_1)$ forms an independent set. Similarly, $V(H_2)$ is an independent set. Due to symmetry $|V(H_1)| = |V(H_2)|$. The above argument shows that H is a bipartite graph.

Claim 4.3.1 We claim that H contains a perfect matching F such that for every edge $e = uv$ in F , $N(u) \cup N(v)$ is a complete bipartite graph. Let $F = \{v_{2\alpha}v_{-2(d+w-\alpha)} \mid 1 \leq \alpha \leq (d + w - 1)\}$.

For $1 \leq \alpha \leq (d + w - 1)$, we have $|2\alpha - [-2(d + w - \alpha)]| = |2(d + w)|$. However, since $n - 2(d + w) = 1 + 2d$, we can deduce that $v_{2\alpha} \sim v_{-2(d+w-\alpha)}$.

For each $\alpha, 1 \leq \alpha \leq (d + w - 1)$, we note that

$$N_H(v_{2\alpha}) = \{v_{-2(d+w-\gamma)} | 1 \leq \gamma \leq \alpha\}; \text{ and}$$

$$N_H(v_{-2(d+w-\alpha)}) = \{v_{2\rho} | \alpha \leq \rho \leq (d + w - 1)\}.$$

To complete the proof of the claim we show that $|2\rho - [-2(d + w - \gamma)]| \in \langle S \rangle$. Since $-2(d + w - \gamma) \equiv (1 + 2d + 2\gamma) \pmod{n}$, it suffices to show that $|(1 + 2d + 2\gamma) - 2\rho| \in S$. We have

$$\begin{aligned} \gamma &\leq \rho \leq (d + w - 1) \\ 2\gamma &\leq 2\rho \leq 2(d + w - 1) \\ 0 &\leq 2\rho - 2\gamma \leq 2(d + w - 1) - 2\gamma \\ 2\gamma - 2(d + w - 1) &\leq 2\gamma - 2\rho \leq 0 \\ 2\gamma + 3 - 2w &\leq (1 + 2d + 2\gamma) - 2\rho \leq 1 + 2d \end{aligned}$$

Since $(1 + 2d + 2\gamma)$ is odd and 2ρ is even, we can deduce that their difference is odd. Therefore, $v_{1+2d+2\gamma} \sim v_{2\rho}$ and $v_{-2(d+w-\gamma)}v_{2\rho}$ is an edge in H completing the proof of Claim 4.3.1.

Therefore, by Corollary 3.2, H is well-covered, and thence by Corollary 2.6, G is well-covered and $\beta(G) = \frac{n-2d-1}{2}$.

(ii) n is even and either $n = 4d + 2$ or $n = 4d + 4$.

Note that $V(H) = \{v_i : i = -2(d+k), -2d, \dots, -4, -2\} \cup \{v_i : i = 2, 4, \dots, 2d, 2(d+k)\}$ for $k = 0$ or 1 . Let v_i and $v_j \in V(H)$ with $v_i \neq v_j$. Since i and j are both even, it follows that their difference is even, hence $j - i \notin \langle S \rangle$. Therefore, $V(H)$ forms an independent set. Hence, G is well-covered and $\beta(G) = \frac{n}{2}$.

Furthermore, G is very well-covered.

We now proceed to prove the ‘only if’ direction.

Case 4.3.1 n is odd and $n \geq 6d + 9$.

Let $I' = \{v_{2d+3}, v_{-(2d+3)}\}$. First, we show that $v_{2d+3} \not\sim v_{-(2d+3)}$. Note that $|(2d+3) - (-(2d+3))| = 4d+6$. Given our assumption that $n \geq 6d+9$, it follows that $n - (4d+6) \geq 2d+3$. Hence, I' is an independent set in G .

Now, let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_i : i = -(2d+1), \dots, -5, -3, -1\} \cup \{v_0\} \cup \{v_i : i = 1, 3, 5, \dots, 2d+1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{2d+1}, v_{-(2d+1)}\}$. Note that $v_{2d+1} \not\sim v_{-(2d+1)}$ since $|(2d+1) - (-(2d+1))| = 4d+2$. Given our assumption that $n \geq 6d+9$, it follows that $n - (4d+2) \geq 2d+7$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 4.3.2 n is even and $n \geq 4d + 6$.

Let $H_1 = G[\{v_i : i = 2, 4, \dots, n-2\}]$ and $H_2 = G[\{v_i : i = 2d+3, 2d+5, \dots, n-(2d+3)\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Since i and j are both even, it follows that their difference is even, hence $j - i \notin \langle S \rangle$. Therefore, $V(H_1)$ forms an independent set. Similarly, $V(H_2)$ is an independent set. Hence, H is a bipartite graph.

Observe that $|V(H_1)| > |V(H_2)|$, and thus H does not contain a perfect matching. Hence, by Proposition 2.5, G is not well-covered. ■

A characterization of the well-covered graphs in Class 6 can now be stated.

Theorem 4.4 *Let n and d be integers with $1 \leq d \leq \frac{n}{4}$. Then $G = C(n, \{1\} \cup \{2, 4, \dots, 2d\})$ is well-covered if and only if one of the following conditions holds:*

- (i) $d \geq 2$ and either $n = 4d$ or $n = 4d + 2$, or
- (ii) $d \geq 2$ and n is odd such that $4d + 1 \leq n \leq 4d + 5$, or
- (iii) $d \geq 2$ and n is even such that $4d + 8 \leq n \leq 6d + 4$, or
- (iv) $d = 1$ and either $4 \leq n \leq 8$ or $n = 11$.

Furthermore, if (i) holds $\beta(G) = 2$; if (ii) holds $\beta(G) = 3$; if (iii) holds $\beta(G) = 4$; and if (iv) holds $\beta(G) = \lfloor \frac{n}{5} \rfloor$.

Proof. For $d = 1$, the theorem follows from Theorem 3.3, hence we assume that $d > 1$. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. Let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$.

- (i) $d \geq 2$ and either $n = 4d$ or $n = 4d + 2$.

For $d = 2$, $C(8, \{1, 2, 4\})$ and $C(10, \{1, 2, 4\})$ are well-covered and $\beta(G) = 2$, hence we may assume that $d \geq 3$.

Set $n = 2(2d + k)$ for $k = 0$ or 1 . Then $V(H) = \{v_i : i = 3, 5, \dots, [2(2d + k) - 3]\}$. Let v_i and $v_j \in V(H)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$\begin{aligned} 3 &\leq i < j \leq 2(2d + k) - 3 \\ 0 &\leq j - i \leq 2(2d + k) - 3 - 3 \\ 0 &\leq j - i \leq 2(2d + k) - 6 \end{aligned}$$

Observe that $j - i \leq 2(2d + k) - 6 < n = 2(2d + k)$. Also note that $n - [2(2d + k) - 6] = 6 \leq \lfloor \frac{n}{2} \rfloor = 2d + k$. Since i and j are both odd, it follows that their

difference is even, and thus v_i and v_j are adjacent. Therefore, H is a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$.

(ii) $d \geq 2$ and n is odd such that $4d + 1 \leq n \leq 4d + 5$.

Let $n = 1 + 2(2d + k)$ for $k = 0, 1$ or 2 . Let $H_1 = G[\{v_i : i = 3, 5, \dots, [2(d + k) - 1]\}]$ and $H_2 = G[\{v_i : i = -[2(d + k) - 1], \dots, -5, -3\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

We claim that H_i is a complete graph for $i = 1$ and 2 . Let v_i and $v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$\begin{aligned} 3 \leq i < j &\leq 2(d + k) - 1 \\ 0 \leq j - i &\leq 2(d + k) - 1 - 3 \\ 0 \leq j - i &\leq 2(d + k) - 4 \end{aligned}$$

Observe that $j - i \leq 2(d + k) - 4 < \lfloor \frac{n}{2} \rfloor = 2d + k$. Since i and j are both odd, it follows that their difference is even. Hence, v_i and v_j are adjacent, therefore H_1 is a complete graph. Similarly, H_2 is a complete graph.

Note that $V(H) \neq \emptyset$, hence $|I \cap V(H)| \neq \emptyset$. Next, consider $v_{2d-1} \in V(H_1)$ and $v_{-(2d-1)} \in V(H_2)$. We claim that $v_{2d-1} \not\sim v_{-(2d-1)}$. Note that $|(2d-1) - (-(2d-1))| = 4d - 2$. Given our assumption that $n = 1 + 2(2d + k)$, it follows that $n - (4d - 2) = 2k + 3$. Since n is odd and $(4d - 2)$ is even, it follows that their difference is odd, and thence $v_{2d-1} \not\sim v_{-(2d-1)}$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 , and hence $|I| = 3$, G is well-covered and $\beta(G) = 3$.

(iii) $d \geq 2$ and n is even such that $4d + 8 \leq n \leq 6d + 4$.

Let $G_1 = G[\{v_i : i = 0, 2, \dots, n - 2\}]$ and $G_2 = G[\{v_i : i = 1, 3, \dots, n - 1\}]$.

Note that $V(G_1)$ together with $V(G_2)$ forms a partition of $V(G)$.

Consider any vertex $v_i \in V(G_1)$. Since i is even, we can deduce that $i + 1$ and $i - 1$ are both odd, and thence v_i is adjacent to exactly two vertices in $V(G_2)$.

Next, we let $H_1 = G[\{v_i : i = (2d + 2), (2d + 4), \dots, n - (2d + 2)\}]$ and $H_2 = G[\{v_i : i = 3, 5, \dots, n - 3\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$. Let $v_i, v_j \in V(H_1)$ with $v_i \neq v_j$. Without loss of generality, assume that $i < j$. Then

$$2d + 2 \leq i < j \leq n - (2d + 2)$$

$$0 \leq j - i \leq n - (2d + 2) - (2d + 2)$$

$$0 \leq j - i \leq n - (4d + 4)$$

Observe that $n - (4d + 4) \leq 2d$, and since i and j are both even, it follows that their difference is even, hence $j - i \in S$. Therefore, H_1 is a complete graph and $|I \cap V(H_1)| = 1$. Hence, $\beta(G_1) = 2$ and $\beta(G)$ is at least two.

Without loss of generality, let $v_k \in I \cap V(H_1)$, and let W be the graph induced by $H \setminus N_H[v_k]$. Next, we consider v_{2d-1} and $v_{4d+5} \in V(W)$. Note that $v_{2d-1} \not\sim v_{4d+5}$ since $|(4d + 5) - (2d - 1)| = 2d + 6 \notin \langle S \rangle$. Therefore, $|I \cap V(W)| = 2$ and $\beta(W) = 2$. Hence, G is well-covered and $\beta(G) = 4$.

We now proceed to prove the ‘only if’ direction.

Case 4.4.1 n is odd and $n \geq 4d + 7$.

Case 4.4.1.1 $n = 4d + 7$.

Let $I_1 = \{v_0, v_{2d+3}, v_{-(2d+1)}\}$. Note that v_0 is adjacent to neither $v_{-(2d+1)}$ nor v_{2d+3} since $|0 - (-(2d + 1))| = 2d + 1 \notin \langle S \rangle$ and $|(2d + 3) - 0| = 2d + 3 \notin \langle S \rangle$. Furthermore,

$v_{2d+3} \not\sim v_{-(2d+1)}$ since $|(2d+3) - (-(2d+1))| = 4d+4 \equiv -3 \pmod{4d+7} \notin \langle S \rangle$. Hence, I_1 is an independent set in G . Next, since v_{2d+3} is adjacent to $v_{(2d+3)-j}$ and $v_{(2d+3)+j}$ for each j in the set S , it follows that v_{2d+3} is adjacent to v_{2d+2} , v_{2d+4} , v_{2d+5} and all v_i 's such that $3 \leq i \leq 2d+1$, where i is odd. Similarly, $v_{-(2d+1)}$ is adjacent to all v_i 's such that $-(2d-1) \leq i \leq -1$, where i is odd. Hence, I_1 is a maximal independent set in G .

Now, let $I_2 = \{v_0, v_3, v_{-(2d-1)}, v_{-(2d+2)}\}$. First, we note that $v_3 \not\sim v_{-(2d-1)}$ since $|3 - (-(2d-1))| = 2d+2 \notin \langle S \rangle$, and $v_{-(2d-1)} \not\sim v_{-(2d+2)}$ since $|-(2d-1) - (-(2d+2))| = 3 \notin S$. Next, observe that v_0 is adjacent to neither $v_{-(2d+2)}$, $v_{-(2d-1)}$ nor v_3 since $|0 - (-(2d+2))| = 2d+2 \notin \langle S \rangle$, $|0 - (-(2d-1))| = 2d-1 \notin \langle S \rangle$ and $|3-0| = 3 \notin S$. Furthermore, $v_3 \not\sim v_{-(2d+2)}$ since $|3 - (-(2d+2))| = 2d+5 \equiv -(2d+2) \pmod{4d+7} \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 4.4.1.2 $n \geq 4d+9$.

Let $I' = \{v_{2d+3}, v_{-(2d+3)}\}$. First, we show that $v_{2d+3} \not\sim v_{-(2d+3)}$. Note that $|(2d+3) - (-(2d+3))| = 4d+6$. Given our assumption that $n \geq 4d+9$, it follows that $n - (4d+6) \geq 3$. Given that n is odd, it follows that $n - (4d+6)$ is odd. Hence, I' is an independent set in G .

Now, let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_i : i = -2d, \dots, -4, -2, -1\} \cup \{v_0\} \cup \{v_i : i = 1, 2, 4, \dots, 2d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{2d}, v_{-2d}\}$. We claim that $v_{2d} \not\sim v_{-2d}$. Note that $|2d - (-2d)| = 4d$. Given our assumption that $n \geq 4d+9$, it follows that $n - 4d \geq 9$. Given that n is odd, it follows that $n - 4d$ is odd. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 4.4.2 n is even and $n = 4d + 4$, $n = 4d + 6$, or $n \geq 6d + 6$.

Case 4.4.2.1 $n = 4d + 4$.

Let $I_1 = \{v_0, v_{2d+1}\}$. We first note that $v_0 \not\sim v_{2d+1}$ since $|(2d+1) - 0| = 2d+1 \notin \langle S \rangle$. Hence, I_1 is an independent set in G . Since v_{2d+1} is adjacent to $v_{(2d+1)-j}$ and $v_{(2d+1)+j}$ for each j in the set S , it follows that v_{2d+1} is adjacent to v_{2d} , v_{2d+2} and all v_i 's such that $1 \leq i \leq 2d-1$ and $2d+3 \leq i \leq 4d+1$, where i is odd. Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_3, v_{2d+2}, v_{-(2d-1)}\}$. First, we note that v_{2d+2} is adjacent to neither v_0 nor v_3 since $|(2d+2) - 0| = 2d+2 \notin \langle S \rangle$ and $|(2d+2) - 3| = 2d-1 \notin \langle S \rangle$. Next, observe that $v_{2d+2} \not\sim v_{-(2d-1)}$ since $|(2d+2) - (-(2d-1))| = 4d+1 \equiv -3 \pmod{4d+4} \notin \langle S \rangle$. Furthermore, from Case 4.4.1.1, we know that v_0 is adjacent to neither $v_{-(2d-1)}$ nor v_3 and $v_3 \not\sim v_{-(2d-1)}$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 4.4.2.2 $n = 4d + 6$.

Let $I_1 = \{v_0, v_{2d+3}\}$. We first note that $v_0 \not\sim v_{2d+3}$ since $|(2d+3) - 0| = 2d+3 \notin \langle S \rangle$. Hence, I_1 is an independent set in G . Since v_{2d+3} is adjacent to $v_{(2d+3)-j}$ and $v_{(2d+3)+j}$ for each j in the set S , it follows that v_{2d+3} is adjacent to v_{2d+2} , v_{2d+4} and all v_i 's such that $3 \leq i \leq 2d+1$ and $2d+5 \leq i \leq 4d+3$, where i is odd. Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_{2d+1}, v_{-3}, v_{-(2d+2)}\}$. First, we note that v_{-3} is adjacent to neither $v_{-(2d+2)}$ nor v_{2d+1} since $|(-3) - (-(2d+2))| = 2d-1 \notin \langle S \rangle$ and $|(2d+1) - (-3)| = 2d+4 \notin \langle S \rangle$. Next, observe that v_0 is adjacent to neither $v_{-(2d+2)}$, v_{-3} nor v_{2d+1} since $|0 - (-(2d+2))| = 2d+2 \notin \langle S \rangle$, $|0 - (-3)| = 3 \notin S$ and $|(2d+1) - 0| = 2d+1 \notin \langle S \rangle$. Furthermore, $v_{2d+1} \not\sim v_{-(2d+2)}$ since $|(2d+1) - (-(2d+2))| = 4d+3 \equiv -3 \pmod{4d+6} \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 4.4.2.3 $n = 6d + 6$.

Let $I' = \{v_{2d+3}, v_{-(2d+1)}, v_{-(2d+4)}\}$. First, we note that v_{2d+3} is adjacent to neither $v_{-(2d+1)}$ nor $v_{-(2d+4)}$ since $|(2d+3) - (-(2d+1))| = 4d+4 \equiv -(2d+2) \pmod{6d+6} \notin \langle S \rangle$ and $|(2d+3) - (-(2d+4))| = 4d+7 \equiv -(2d-1) \pmod{6d+6} \notin \langle S \rangle$. Furthermore, $v_{-(2d+1)} \not\sim v_{-(2d+4)}$ since $|-(2d+1) - (-(2d+4))| = 3 \notin S$. Hence, I_1 is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-2}\} \cup \{v_i : i = 0, 1, 2, \dots, 2d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{2d}, v_{-2}\}$. Note that $v_{2d} \not\sim v_{-2}$ since $|2d - (-2)| = 2d + 2 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 4.4.2.4 $n \geq 6d + 8$.

Let $I' = \{v_{2d+3}, v_{-(2d+3)}\}$. First, we show that $v_{2d+3} \not\sim v_{-(2d+3)}$. Note that $|(2d+3) - (-(2d+3))| = 4d+6$. Given our assumption that $n \geq 6d+8$, it follows that $n - (4d+6) \geq 2d+2$. Hence, I' is an independent set in G .

Now, let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_i : i = -2d, \dots, -4, -2, -1\} \cup \{v_0\} \cup \{v_i : i = 1, 2, 4, \dots, 2d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{2d}, v_{-2d}\}$. We claim that $v_{2d} \not\sim v_{-2d}$. Note that $|2d - (-2d)| = 4d$. Given our assumption that $n \geq 6d+8$, it follows that $n - 4d \geq 2d+8$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

■

Chapter 5

Characterization of Well-Covered Graphs in Classes 7, 8, 9, and 10

In this chapter, we investigate the class of circulant graphs on n vertices with a generating set $S - A$, where A is one of $\{1\}$, $\{2\}$, $\{1, 2\}$, or $\{2, 3\}$ and $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. To determine if these classes are well-covered we apply the following lemma.

Lemma 5.1 *Let w_0 be a vertex of a graph G and $N[w_0] = \{w_0, w_1, w_2, w_3, \dots, w_k\}$. If $G \setminus N[w_i]$ is well-covered for each i and if $\beta(G \setminus N[w_i]) = \beta(G \setminus N[w_j]) \forall i, j \in \{0, 1, \dots, k\}$ then G is well-covered with $\beta(G) = \beta(G \setminus N[w_0]) + 1$.*

Proof. By Corollary 2.6, G is well-covered with $\beta(G) = \beta(G \setminus N[w_0]) + 1$. ■

A characterization of the well-covered graphs in Class 7 can now be stated.

Theorem 5.2 *Let n and d be integers with $2 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{2, 3, \dots, d\})$ is well-covered if and only if one of the following conditions holds:*

- (i) $2d \leq n \leq 2d + 2$, or
- (ii) $n = 2d + 3$, or
- (iii) $2d + 4 \leq n \leq 3d + 2$, or
- (iv) $n = 3d + 4$, or
- (v) $n = 4d + 6$.

Furthermore, if (i) holds $\beta(G) = 2$; if (ii) holds $\beta(G) = 3$; if (iii) or (iv) holds $\beta(G) = 4$; and if (v) holds $\beta(G) = 6$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n-1\}$. First, we prove the ‘if’ direction. In each case let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Then $V(H) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{-1}\} \cup \{v_1\} \cup \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. By Corollary 2.6, it suffices to show that in Case (i) H is well-covered with $\beta(H) = 1$; in Case (ii) H is well-covered with $\beta(H) = 2$; in Case (iii) or (iv) H is well-covered with $\beta(H) = 3$; and in Case (v) H is well-covered with $\beta(H) = 5$.

(i) $2d \leq n \leq 2d+2$.

Observe that for $n = 2d$ or $2d+1$, the theorem follows as a consequence of Theorem 3.4. Furthermore, $\beta(G) = 2$.

Next, we consider $n = 2d+2$. In this case $V(H) = \{v_{-1}, v_1, v_{d+1}\}$. Note that v_1 is adjacent to both v_{-1} and v_{d+1} since $|1 - (-1)| = 2 \in S$ and $|(d+1) - 1| = d \in S$. Furthermore, $v_{-1} \sim v_{d+1}$ since $|(d+1) - (-1)| = d+2 \equiv -d \pmod{2d+2} \in \langle S \rangle$. Therefore, H is a complete graph and $|I \cap V(H)| = 1$. Hence, G is well-covered and $\beta(G) = 2$.

(ii) $n = 2d+3$.

In this case $V(H) = \{v_{-(d+1)}, v_{-1}, v_1, v_{d+1}\}$. Observe that v_{d+1} is adjacent to neither $v_{-(d+1)}$ nor v_{-1} since $|(d+1) - (-(d+1))| = 2d+2 \equiv -1 \pmod{2d+3} \notin \langle S \rangle$ and $|(d+1) - (-1)| = d+2 \equiv -(d+1) \pmod{2d+3} \notin \langle S \rangle$. From Case (i), we know that v_1 is adjacent to both v_{-1} and v_{d+1} . By symmetry, we can also deduce that $v_{-(d+1)} \not\sim v_1$ and $v_{-(d+1)} \sim v_{-1}$. Hence, H is isomorphic to P_4 and $|I \cap V(H)| = 2$. Therefore, G is well-covered and $\beta(G) = 3$.

(iii) $2d + 4 \leq n \leq 3d + 2$.

We consider the following five cases:

Case 5.2.1 $d \geq 2$ and $n = 2d + 4$.

Let $H' = G[\{v_{-(d+1)}, v_{-1}, v_1, v_{d+1}\}]$ and $H'' = G[\{v_{d+2}\}]$. Note that $V(H')$ together with $V(H'')$ forms a partition of $V(H)$.

We claim that no vertex in H' is adjacent to a vertex in H'' . Note that v_{d+2} is adjacent to neither v_1 nor v_{d+1} since $|(d+2) - 1| = d+1 \notin \langle S \rangle$ and $|(d+2) - (d+1)| = 1 \notin S$. Furthermore, v_{d+2} is adjacent to neither $v_{-(d+1)}$ nor v_{-1} since $|(d+2) - (-(d+1))| = 2d+3 \equiv -1 \pmod{2d+4} \notin \langle S \rangle$ and $|(d+2) - (-1)| = d+3 \equiv -(d+1) \pmod{2d+4} \notin \langle S \rangle$. Hence, $\beta(H'') = 1$.

We now show that H' is isomorphic to C_4 . First, we note that $v_{d+1} \sim v_{-(d+1)}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -2 \pmod{2d+4} \in \langle S \rangle$. Next, observe that $v_{-(d+1)} \not\sim v_1$ since $|1 - (-(d+1))| = d+2 \notin \langle S \rangle$. From Case (i), we know that v_1 is adjacent to both v_{-1} and v_{d+1} . By symmetry, we can also deduce that $v_{-(d+1)} \sim v_{-1}$ and $v_{d+1} \not\sim v_{-1}$. Hence, H' is isomorphic to C_4 , and thus is well-covered with $\beta(H') = 2$. Therefore, G is well-covered and $\beta(G) = 4$ concluding the proof of Case 5.2.1.

Case 5.2.2 $d \geq 3$ and $n = 2d + 5$.

In this case $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_1, v_{d+1}, v_{d+2}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-1}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.2.2.1 $w = v_{-1}$ or $w = v_1$.

By symmetry, we need only examine $w = v_1$. Let $H'_w = H[\{v_{-(d+2)}\}]$ and $H''_w = H[\{v_{-(d+1)}, v_{d+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -1 \pmod{2d+5} \notin \langle S \rangle$. Furthermore, $v_{-(d+1)} \not\sim v_{-(d+2)}$ since $|-(d+1) - (-(d+2))| = 1 \notin S$.

We now note that $v_{-(d+1)} \sim v_{d+2}$ since $|(d+2) - (-(d+1))| = 2d+3 \equiv -2 \pmod{2d+5} \in \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.2.2.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-1}, v_{d+2}\}$. Note that $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.2.2.

Case 5.2.3 $d \geq 4$ and $n = 2d + 6$.

In this case $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_1, v_{d+1}, v_{d+2}, v_{d+3}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-1}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.2.3.1 $w = v_{-1}$ or $w = v_1$.

By symmetry, we need only examine $w = v_1$. We first note that $V(H_w) = \{v_{-(d+2)}, v_{-(d+1)}, v_{d+2}, v_{d+3}\}$.

We claim that H_w is isomorphic to P_4 . First, we note that v_{d+2} is adjacent to both $v_{-(d+2)}$ and $v_{-(d+1)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -2 \pmod{2d+6} \in \langle S \rangle$ and $|(d+2) - (-(d+1))| = 2d+3 \equiv -3 \pmod{2d+6} \in \langle S \rangle$. Next, observe that $v_{d+3} \not\sim v_{-(d+2)}$ since $|(d+3) - (-(d+2))| = 2d+5 \equiv -1 \pmod{2d+6} \notin \langle S \rangle$. Furthermore, $v_{d+2} \not\sim v_{d+3}$ since $|(d+3) - (d+2)| = 1 \notin S$, $v_{-(d+2)} \not\sim v_{-(d+1)}$ since $|-(d+1) - (-(d+2))| = 1 \notin S$ and $v_{d+3} \sim v_{-(d+1)}$ since $|(d+3) - (-(d+1))| = 2d+4 \equiv -2 \pmod{2d+6} \in \langle S \rangle$. Hence, H_w is isomorphic to P_4 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.2.3.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-1}, v_{d+2}\}$. Note that $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.2.3.

Case 5.2.4 $d \geq 5$ and $n = 2d + 7$.

In this case $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_1, v_{d+1}, v_{d+2}, v_{d+3}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-1}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.2.4.1 $w = v_{-1}$ or $w = v_1$.

By symmetry, we need only examine $w = v_1$. We first note that $V(H_w) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+2}, v_{d+3}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{-(d+3)}$. Since $v_{-(d+3)}$ is adjacent to $v_{-(d+3)+j}$ and $v_{-(d+3)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+3)}] =$

$\{v_{-(d+3)}, v_{-(d+1)}, v_{d+2}\}$. We will show for each $u \in N_{H_w}[v_{-(d+3)}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.2.4.1.1 $u = v_{-(d+3)}$.

In this case $V(H_u) = \{v_{-(d+2)}, v_{d+3}\}$. Note that $v_{d+3} \sim v_{-(d+2)}$ since $|(d+3) - (-(d+2))| = 2d+5 \equiv -2 \pmod{2d+7} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.4.1.2 $u = v_{d+2}$.

In this case $V(H_u) = \{v_{d+3}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.4.1.3 $u = v_{-(d+1)}$.

In this case $V(H_u) = \{v_{-(d+2)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.2.4.1.

Case 5.2.4.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-1}, v_{d+2}\}$. Note that $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.2.4.

Case 5.2.5 $d \geq 6$ and $2d+8 \leq n \leq 3d+2$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-1}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.2.5.1 $w = v_{-1}$ or $w = v_1$.

By symmetry, we need only examine $w = v_1$. We first note that $V(H_w) = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_i: d+2 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{d+2}$. Since v_{d+2} is adjacent to $v_{(d+2)+j}$ and $v_{(d+2)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{d+2}] = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{d+2}\} \cup \{v_i: d+4 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We will show for each $u \in N_{H_w}[v_{d+2}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.2.5.1.1 $u = v_{d+2}$.

In this case $V(H_u) = \{v_{d+3}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.5.1.2 $u = v_k$ or $u = v_{-k}$ for $d+4 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By symmetry, we need only examine $u = v_k$. Since 1 is not in S , we can deduce that v_k is adjacent to neither v_{k-1} nor v_{k+1} , and thence $V(H_u) = \{v_{k-1}, v_{k+1}\}$. Note that $v_{k+1} \sim v_{k-1}$ since $|(k+1) - (k-1)| = 2 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.5.1.3 $u = v_{-(d+3)}$.

In this case $V(H_u) = \{v_{-(d+4)}, v_{-(d+2)}\}$. Note that $v_{-(d+4)} \sim v_{-(d+2)}$ since $|-(d+2) - (-(d+4))| = 2 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.5.1.4 $u = v_{-(d+2)}$.

In this case $V(H_u) = \{v_{-(d+3)}, v_{-(d+1)}\}$. Note that $v_{-(d+1)} \sim v_{-(d+3)}$ since $|-(d+1) - (-(d+3))| = 2 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.5.1.5 $u = v_{-(d+1)}$.

In this case $V(H_u) = \{v_{-(d+2)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.2.5.1.

Case 5.2.5.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-1}, v_{d+2}\}$. Note that $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.2.5.

(iv) $n = 3d + 4$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-1}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.2.6 $w = v_{-1}$ or $w = v_1$.

By symmetry, we need only examine $w = v_1$. We first note that $V(H_w) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_i : d+2 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{d+2}$. Since v_{d+2} is adjacent to $v_{(d+2)+j}$ and $v_{(d+2)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{d+2}] = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+2)\} \cup \{v_{d+2}\} \cup \{v_i : d+4 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We will show for each $u \in N_{H_w}[v_{d+2}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.2.6.1 $u = v_{d+2}$.

In this case $V(H_u) = \{v_{-(d+1)}, v_{d+3}\}$. Note that $v_{d+3} \sim v_{-(d+1)}$ since $|(d+3) - (-(d+1))| = 2d+4 \equiv -d \pmod{3d+4} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.2.6.2 $u = v_k$ for $d + 4 \leq k \leq 2d + 2$.

Since 1 is not in S , we can deduce that v_k is adjacent to neither v_{k-1} nor v_{k+1} , and thence $V(H_u) = \{v_{k-1}, v_{k+1}\}$. Note that $v_{k+1} \sim v_{k-1}$ since $|(k+1) - (k-1)| = 2 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.2.6.

Case 5.2.7 $w = v_{d+1}$.

Let $H'_w = H[\{v_{-(d+1)}, v_{-1}\}]$ and $H''_w = H[\{v_{-(d+2)}, v_{d+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that v_{-1} is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} since $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$ and $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Next, we note that $v_{-(d+1)}$ is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} since $|-(d+1) - (-(d+2))| = 1 \notin S$ and $|(d+2) - (-(d+1))| = 2d+3 \equiv -(d+1) \pmod{3d+4} \notin \langle S \rangle$.

We now note that $v_{d+2} \sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -d \pmod{3d+4} \in \langle S \rangle$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_{-(d+1)} \sim v_{-1}$ since $|(-1) - (-(d+1))| = d \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case (iv).

(v) $n = 4d + 6$.

First, we note that $C(14, \{2\})$, $C(18, \{2, 3\})$, and $C(22, \{2, 3, 4\})$ are well-covered and $\beta(G) = 6$.

Next, we consider the case where $d \geq 5$. Note that $V(H) = \{v_i: -(2d+2) \leq i \leq -(d+1)\} \cup \{v_{-1}\} \cup \{v_1\} \cup \{v_i: d+1 \leq i \leq 2d+2\} \cup \{v_{2d+3}\}$. To show

that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{2d+3}$. Since v_{2d+3} is adjacent to $v_{(2d+3)+j}$ and $v_{(2d+3)-j}$ for each j in the set S , it follows that $N_H[v_{2d+3}] = \{v_i: -(2d+1) \leq i \leq -(d+3)\} \cup \{v_i: d+3 \leq i \leq 2d+1\} \cup \{v_{2d+3}\}$. We will show for each $w \in N_H[v_{2d+3}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.2.8 $w = v_{2d+3}$.

Let $H'_w = H[\{v_{-(d+1)}, v_{-1}, v_1, v_{d+1}\}]$ and $H''_w = H[\{v_{-(2d+2)}, v_{-(d+2)}, v_{d+2}, v_{2d+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that v_1 is adjacent to neither $v_{-(2d+2)}, v_{-(d+2)}, v_{d+2}$ nor v_{2d+2} since $|1 - (-(2d+2))| = 2d+3 \notin \langle S \rangle$, $|1 - (-(d+2))| = d+3 \notin \langle S \rangle$, $|(d+2) - 1| = d+1 \notin \langle S \rangle$ and $|(2d+2) - 1| = 2d+1 \notin \langle S \rangle$. Next, we note that v_{d+1} is adjacent to neither $v_{-(d+2)}, v_{d+2}$ nor v_{2d+2} since $|(d+1) - (-(d+2))| = 2d+3 \notin \langle S \rangle$, $|(d+2) - (d+1)| = 1 \notin S$ and $|(2d+2) - (d+1)| = d+1 \notin \langle S \rangle$. Furthermore, $v_{d+1} \not\sim v_{-(2d+2)}$ since $|(d+1) - (-(2d+2))| = 3d+3 \equiv -(d+3) \pmod{4d+6} \notin \langle S \rangle$. By symmetry, we can also deduce that v_{-1} is adjacent to neither $v_{-(2d+2)}, v_{-(d+2)}, v_{d+2}$ nor v_{2d+2} ; and $v_{-(d+1)}$ is adjacent to neither $v_{-(2d+2)}, v_{-(d+2)}, v_{d+2}$ nor v_{2d+2} .

We now show that H'_w is isomorphic to P_4 . Observe that v_1 is adjacent to both v_{-1} and v_{d+1} since $|1 - (-1)| = 2 \in S$ and $|(d+1) - 1| = d \in S$. Next, we note that $v_{-(d+1)}$ is adjacent to neither v_1 nor v_{d+1} since $|(1) - (-(d+1))| = d+2 \notin \langle S \rangle$ and $|(d+1) - (-(d+1))| = 2d+2 \notin \langle S \rangle$. By symmetry, we can also deduce that $v_{-(d+1)} \sim v_{-1}$ and $v_{d+1} \not\sim v_{-1}$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Finally, we show that H''_w is isomorphic to P_4 . Observe that v_{2d+2} is adjacent to

both v_{d+2} and $v_{-(2d+2)}$ since $|(2d+2) - (d+2)| = d \in S$ and $|(2d+2) - (-(2d+2))| = 4d+4 \equiv -2 \pmod{4d+6} \in \langle S \rangle$. Next, we note that $v_{-(2d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(2d+2))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$, and $v_{-(d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+2))| = 2d+4 \notin \langle S \rangle$. By symmetry, we can also deduce that $v_{-(2d+2)} \sim v_{-(d+2)}$ and $v_{2d+2} \not\sim v_{-(d+2)}$. Hence, H_w'' is isomorphic to P_4 , and thus is well-covered with $\beta(H_w'') = 2$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.2.9 $w = v_{d+3}$ or $w = v_{-(d+3)}$.

By symmetry, we need only examine $w = v_{d+3}$. Note that $V(H_w) = \{v_i : -(2d+2) \leq i \leq -(d+1)\} \cup \{v_{-1}, v_1, v_{d+2}, v_{d+4}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{-(2d+1)}$. Since $v_{-(2d+1)}$ is adjacent to $v_{-(2d+1)+j}$ and $v_{-(2d+1)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(2d+1)}] = \{v_{-(2d+1)}\} \cup \{v_i : -(2d-1) \leq i \leq -(d+1)\}$. We will show for each $u \in N_{H_w}[v_{-(2d+1)}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 3$.

Case 5.2.9.1 $u = v_{-(2d+1)}$.

Let $H_u' = H_w[\{v_{-(2d+2)}, v_{-2d}, v_{d+2}, v_{d+4}\}]$ and $H_u'' = H_w[\{v_{-1}, v_1\}]$. Note that $V(H_u')$ together with $V(H_u'')$ forms a partition of $V(H_u)$.

We claim that no vertex in H_u' is adjacent to a vertex in H_u'' . Observe that v_1 is adjacent to neither $v_{-(2d+2)}, v_{-2d}, v_{d+2}$ nor v_{d+4} since $|1 - (-(2d+2))| = 2d+3 \notin \langle S \rangle$, $|1 - (-2d)| = 2d+1 \notin \langle S \rangle$, $|(d+2) - 1| = d+1 \notin \langle S \rangle$ and $|(d+4) - 1| = d+3 \notin \langle S \rangle$. Also note that v_{-1} is adjacent to neither $v_{-(2d+2)}, v_{-2d}, v_{d+2}$ nor v_{d+4} since $|(-1) - (-(2d+2))| = 2d+1 \notin \langle S \rangle$, $|(-1) - (-2d)| = 2d-1 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$.

Next, we show that H_u' is isomorphic to P_4 . Observe that $v_{d+4} \sim v_{-(2d+2)}$ since $|(d+4) - (-(2d+2))| = 3d+6 \equiv -d \pmod{4d+6} \in \langle S \rangle$, and $v_{d+4} \not\sim v_{-2d}$

since $|(d+4) - (-2d)| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$. Also note that v_{d+2} is adjacent to neither $v_{-(2d+2)}$ nor v_{-2d} since $|(d+2) - (-(2d+2))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$ and $|(d+2) - (-2d)| = 3d+2 \equiv -(d+4) \pmod{4d+6} \notin \langle S \rangle$. Furthermore, $v_{d+2} \sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \in S$, and $v_{-2d} \sim v_{-(2d+2)}$ since $|(-2d) - (-(2d+2))| = 2 \in S$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that $v_{-1} \sim v_1$ since $|1 - (-1)| = 2 \in S$, hence H''_u is well-covered with $\beta(H''_u) = 1$. Therefore, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.2.9.2 $u = v_k$ for $-(2d-1) \leq k \leq -(d+3)$.

Since 1 is not in S , we can deduce that v_k is adjacent to neither v_{k-1} nor v_{k+1} . Next, we note that $v_k \not\sim v_1$ since $d+4 \leq 1-k \leq 2d$ and $v_k \not\sim v_{-1}$ since $d+2 \leq -1-k \leq 2d-2$. Furthermore, $v_k \not\sim v_{d+2}$ since $2d+5 \leq (d+2)-k \leq 3d+1$ and $v_k \not\sim v_{d+4}$ since $2d+7 \leq (d+4)-k \leq 3d+3$.

Let $H'_u = H_w[\{v_1, v_{-1}\}]$, $H''_u = H_w[\{v_{k-1}, v_{k+1}\}]$ and $H'''_u = H_w[\{v_{d+2}, v_{d+4}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ and $V(H'''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in one of the graphs H'_u , H''_u and H'''_u is adjacent to a vertex in either of the other two graphs. Observe that v_1 is adjacent to neither v_{k-1} nor v_{k+1} since $d+5 \leq 1-(k-1) \leq 2d+1$ and $d+3 \leq 1-(k+1) \leq 2d-1$. Similarly, v_{-1} is adjacent to neither v_{k-1} nor v_{k+1} since $d+3 \leq (-1)-(k-1) \leq 2d-1$ and $d+1 \leq (-1)-(k+1) \leq 2d-3$. Next, we note that v_1 is adjacent to neither v_{d+2} nor v_{d+4} since $|(d+2) - 1| = d+1 \notin \langle S \rangle$ and $|(d+4) - 1| = d+3 \notin \langle S \rangle$. Similarly, v_{-1} is adjacent to neither v_{d+2} nor v_{d+4} since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Furthermore, v_{d+2} is adjacent to neither v_{k-1} nor v_{k+1} since $2d+6 \leq (d+2) - (k-1) \leq 3d+2$ and $2d+4 \leq (d+2) - (k+1) \leq 3d$; and v_{d+4} is adjacent to neither v_{k-1} nor v_{k+1}

since $2d+8 \leq (d+4) - (k-1) \leq 3d+4$ and $2d+6 \leq (d+4) - (k+1) \leq 3d+2$.

Finally, we note that $v_{-1} \sim v_1$ since $|1 - (-1)| = 2 \in S$; $v_{k+1} \sim v_{k-1}$ since $|(k+1) - (k-1)| = 2 \in S$; and $v_{d+2} \sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \in S$. Hence, H'_u , H''_u and H'''_u are well-covered with $\beta(H'_u) = \beta(H''_u) = \beta(H'''_u) = 1$. Therefore, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.2.9.3 $u = v_{-(d+2)}$.

Let $H'_u = H_w[\{v_{-(d+3)}, v_{-(d+1)}, v_{-1}, v_1\}]$ and $H''_u = H_w[\{v_{d+2}, v_{d+4}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in H'_u is adjacent to a vertex in H''_u . Observe that v_{d+2} is adjacent to neither $v_{-(d+3)}$, $v_{-(d+1)}$, v_{-1} nor v_1 since $|(d+2) - (-(d+3))| = 2d+5 \notin \langle S \rangle$, $|(d+2) - (-(d+1))| = 2d+3 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+2) - 1| = d+1 \notin \langle S \rangle$. Next, we note that v_{d+4} is adjacent to neither $v_{-(d+3)}$, $v_{-(d+1)}$, v_{-1} nor v_1 since $|(d+4) - (-(d+3))| = 2d+7 \notin \langle S \rangle$, $|(d+4) - (-(d+1))| = 2d+5 \notin \langle S \rangle$, $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$ and $|(d+4) - 1| = d+3 \notin \langle S \rangle$.

Next, we show that H'_u is isomorphic to P_4 . Observe that v_{-1} is adjacent to both $v_{-(d+1)}$ and v_1 since $|(-1) - (-(d+1))| = d \in S$ and $|1 - (-1)| = 2 \in S$. Also note that $v_{-(d+3)} \sim v_{-(d+1)}$ since $|-(d+1) - (-(d+3))| = 2 \in S$, and $v_{-1} \not\sim v_{-(d+3)}$ since $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$. Furthermore, v_1 is adjacent to neither $v_{-(d+3)}$ nor $v_{-(d+1)}$ since $|1 - (-(d+3))| = d+4 \notin \langle S \rangle$ and $|1 - (-(d+1))| = d+2 \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that $v_{d+4} \sim v_{d+2}$ since $|(d+4) - (d+2)| = 2 \in S$, hence H''_u is well-covered with $\beta(H''_u) = 1$. Therefore, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.2.9.4 $u = v_{-(d+1)}$.

Let $H'_u = H_w[\{v_{-(2d+2)}, v_{-(d+2)}, v_{d+2}, v_{d+4}\}]$ and $H''_u = H_w[\{v_1\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in H'_u is adjacent to a vertex in H''_u . Note that v_1 is adjacent to neither $v_{-(2d+2)}$, $v_{-(d+2)}$, v_{d+2} nor v_{d+4} since $|1 - (-(2d+2))| = 2d+3 \notin \langle S \rangle$, $|1 - (-(d+2))| = d+3 \notin \langle S \rangle$, $|(d+2) - 1| = d+1 \notin \langle S \rangle$ and $|(d+4) - 1| = d+3 \notin \langle S \rangle$.

Next, we show that H'_u is isomorphic to P_4 . Observe that $v_{d+4} \sim v_{-(2d+2)}$ since $|(d+4) - (-(2d+2))| = 3d+6 \equiv -d \pmod{4d+6} \in \langle S \rangle$, and $v_{d+2} \not\sim v_{-(2d+2)}$ since $|(d+2) - (-(2d+2))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$. Also note that $v_{d+2} \sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \in S$, and $v_{-(2d+2)} \sim v_{-(d+2)}$ since $|-(d+2) - (-(2d+2))| = d \in S$. Furthermore, $v_{-(d+2)}$ is adjacent to neither v_{d+2} nor v_{d+4} since $|(d+2) - (-(d+2))| = 2d+4 \notin \langle S \rangle$ and $|(d+4) - (-(d+2))| = 2d+6 \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that H''_u is well-covered with $\beta(H''_u) = 1$. Therefore, H_u is well-covered with $\beta(H_u) = 3$.

This concludes Case 5.2.9.

Case 5.2.10 $w = v_k$ or $w = v_{-k}$ for $d+4 \leq k \leq 2d-1$.

By symmetry, we need only examine $w = v_k$. Note that $V(H_w) = \{v_{-1}, v_1, v_{k-1}, v_{k+1}\} \cup \{v_i : k+d+1 \leq i \leq 3d+5\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $x_0 = v_{k+d+2}$. Since v_{k+d+2} is adjacent to $v_{(k+d+2)+j}$ and $v_{(k+d+2)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{(k+d+2)}] = \{v_{k+d+2}\} \cup \{v_i : k+d+4 \leq i \leq 3d+5\}$. We will show for each $x \in N_{H_w}[v_{k+d+2}]$

that $H_x = H_w \setminus N_{H_w}[x]$ is well-covered with $\beta(H_x) = 3$.

Case 5.2.10.1 $x = v_{k+d+2}$.

Let $H'_x = H_w[\{v_{-1}, v_1\}]$ and $H''_x = H_w[\{v_{k-1}, v_{k+1}, v_{k+d+1}, v_{k+d+3}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ forms a partition of $V(H_x)$.

We claim that no vertex in H'_x is adjacent to a vertex in H''_x . First, we consider v_1 . Observe that $v_1 \not\sim v_{k+1}$ since $|(k+1) - 1| = k \notin \langle S \rangle$. Also note that $|(k-1) - 1| = k-2$, $|(k+d+1) - 1| = k+d$ and $|(k+d+3) - 1| = k+d+2$. Since $d+4 \leq k \leq 2d-1$, it follows that $d+2 \leq k-2 \leq 2d-3$, $2d+4 \leq k+d \leq 3d-1$ and $2d+6 \leq k+d+2 \leq 3d+1$. Given our assumption that $n = 4d+6$, it follows that $d+7 \leq n - (k+d) \leq 2d+2$ and $d+5 \leq n - (k+d+2) \leq 2d$. Hence, v_1 is adjacent to neither v_{k-1} , v_{k+d+1} nor v_{k+d+3} . Next, we consider v_{-1} . Note that $|(k+1) - (-1)| = k+2$ and $|(k+d+3) - (-1)| = k+d+4$. Since $d+4 \leq k \leq 2d-1$, it follows that $d+6 \leq k+2 \leq 2d+1$ and $2d+8 \leq k+d+4 \leq 3d+3$. Given our assumption that $n = 4d+6$, it follows that $d+3 \leq n - (k+d+4) \leq 2d-2$. Hence, v_{-1} is adjacent to neither v_{k+1} nor v_{k+d+3} . Furthermore, $v_{-1} \not\sim v_{k-1}$ since $|(k-1) - (-1)| = k \notin \langle S \rangle$, and $v_{-1} \not\sim v_{k+d+1}$ since $|(k+d+1) - (-1)| = k+d+2 \notin \langle S \rangle$.

Next, we show that H''_x is isomorphic to P_4 . Observe that $v_{k-1} \sim v_{k+1}$ since $|(k+1) - (k-1)| = 2 \in S$, and $v_{k+1} \sim v_{k+d+1}$ since $|(k+d+1) - (k+1)| = d \in S$. Also note that $v_{k+d+1} \sim v_{k+d+3}$ since $|(k+d+3) - (k+d+1)| = 2 \in S$, and $v_{k+d+3} \not\sim v_{k-1}$ since $|(k+d+3) - (k-1)| = d+4 \notin \langle S \rangle$. Furthermore, $v_{k+d+3} \not\sim v_{k+1}$ since $|(k+d+3) - (k+1)| = d+2 \notin \langle S \rangle$, and $v_{k+d+1} \not\sim v_{k-1}$ since $|(k+d+1) - (k-1)| = d+2 \notin \langle S \rangle$. Hence, H''_x is isomorphic to P_4 , and thus is well-covered with $\beta(H''_x) = 2$.

Finally, we note that $v_{-1} \sim v_1$ since $|1 - (-1)| = 2 \in S$, hence H'_x is well-covered

with $\beta(H'_x) = 1$. Therefore, H_x is well-covered with $\beta(H_x) = 3$.

Case 5.2.10.2 $x = v_u$ for $k + d + 4 \leq u \leq 3d + 3$.

Let $H'_x = H_w[\{v_{-1}, v_1\}]$, $H''_x = H_w[\{v_{k-1}, v_{k+1}\}]$ and $H'''_x = H_w[\{v_{u-1}, v_{u+1}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ and $V(H'''_x)$ forms a partition of $V(H_x)$.

We claim that no vertex in H'_x is adjacent to a vertex in H'''_x . First, we consider v_1 . Note that $|(u-1)-1| = u-2$ and $|(u+1)-1| = u$. Since $k+d+4 \leq u \leq 3d+3$, it follows that $k+d+2 \leq u-2 \leq 3d+1$. Given our assumption that $n = 4d+6$, it follows that $d+5 \leq n-(u-2) \leq 3d+4-k$ and $d+3 \leq n-u \leq 3d+2-k$. Furthermore, since $-k \leq -(d+4)$, it follows that $3d+2-k \leq 2d-2$ and $3d+4-k \leq 2d$. Hence, v_1 is adjacent to neither v_{u-1} nor v_{u+1} . Next, we consider v_{-1} . Note that $|(u-1)-(-1)| = u$ and $|(u+1)-(-1)| = u+2$. Since $k+d+4 \leq u \leq 3d+3$, it follows that $k+d+6 \leq u+2 \leq 3d+5$. Given our assumption that $n = 4d+6$, it follows that $d+1 \leq n-(u+2) \leq 3d-k$. Furthermore, since $-k \leq -(d+4)$, it follows that $3d-k \leq 2d-4$. Hence, v_{-1} is adjacent to neither v_{u-1} nor v_{u+1} .

Next, we claim that no vertex in H''_x is adjacent to a vertex in H'''_x . Observe that $v_1 \not\sim v_{k+1}$ since $|(k+1)-1| = k \notin \langle S \rangle$, and $v_{-1} \not\sim v_{k-1}$ since $|(k-1)-(-1)| = k \notin \langle S \rangle$. We now show that $v_{-1} \not\sim v_{k+1}$. Note that $|(k+1)-(-1)| = k+2$. Since $d+4 \leq k \leq 2d-1$, it follows that $d+6 \leq k+2 \leq 2d+1$. Similarly, $v_1 \not\sim v_{k-1}$ since $|(k-1)-1| = k+2 \notin \langle S \rangle$.

We also claim that no vertex in H''_x is adjacent to a vertex in H'''_x . First, we consider v_{k-1} . Note that $|(u-1)-(k-1)| = u-k$ and $|(u+1)-(k-1)| = u-k+2$. Since $k+d+4 \leq u \leq 3d+3$ and $d+4 \leq k \leq 2d-1$, it follows that $k-d+5 \leq u-k \leq 2d-1$ and $k-d+7 \leq u-k+2 \leq 2d+1$. Furthermore, $k-d+5 \leq d+4$ and $k-d+7 \leq d+6$. Hence, v_{k-1} is adjacent to neither v_{u-1}

nor v_{u+1} . Next, we consider v_{k+1} . Note that $|(u-1)-(k+1)| = u-k-2$ and $|(u+1)-(k+1)| = u-k$. Since $k+d+4 \leq u \leq 3d+3$ and $d+4 \leq k \leq 2d-1$, it follows that $k-d+3 \leq u-k-2 \leq 2d-3$. Furthermore, $k-d+3 \leq d+2$. Hence, v_{k+1} is adjacent to neither v_{u-1} nor v_{u+1} .

To conclude $v_{-1} \sim v_1$ since $|1-(-1)| = 2 \in S$; $v_{k-1} \sim v_{k+1}$ since $|(k+1)-(k-1)| = 2 \in S$; and $v_{u-1} \sim v_{u+1}$ since $|(u+1)-(u-1)| = 2 \in S$. Hence, H'_x , H''_x and H'''_x are well-covered with $\beta(H'_x) = \beta(H''_x) = \beta(H'''_x) = 1$. Therefore, $V(H_x)$ is well-covered with $\beta(H_x) = 3$.

Case 5.2.10.3 $x = v_{3d+4}$.

Let $H'_x = H_w[\{v_{-1}, v_1, v_{3d+3}, v_{3d+5}\}]$ and $H''_x = H_w[\{v_{k-1}, v_{k+1}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ forms a partition of $V(H_x)$.

We claim that no vertex in H'_x is adjacent to a vertex in H''_x . First, we consider v_{3d+3} . Note that $|(3d+3)-(k-1)| = 3d-k+4$ and $|(3d+3)-(k+1)| = 3d-k+2$. Since $d+4 \leq k \leq 2d-1$, it follows that $d+5 \leq 3d-k+4 \leq 2d$ and $d+4 \leq 3d-k+3 \leq 2d-1$. Hence, v_{3d+3} is adjacent to neither v_{k-1} nor v_{k+1} . Next, we consider v_{3d+5} . Note that $|(3d+5)-(k-1)| = 3d-k+6$ and $|(3d+5)-(k+1)| = 3d-k+4$. Since $d+4 \leq k \leq 2d-1$, it follows that $d+7 \leq 3d-k+6 \leq 2d+2$. Hence, v_{3d+5} is adjacent to neither v_{k-1} nor v_{k+1} . Furthermore, from Case 5.2.10.1, we know that v_{k-1} is adjacent to neither v_{-1} nor v_1 ; and v_{k+1} is adjacent to neither v_{-1} nor v_1 .

Next, we show that H'_x is isomorphic to P_4 . Observe that $v_{3d+3} \sim v_{3d+5}$ since $|(3d+5)-(3d+3)| = 2 \in S$, and $v_1 \sim v_{-1}$ since $|1-(-1)| = 2 \in S$. Next, we note that $v_{3d+5} \sim v_{-1}$ since $|(3d+5)-(-1)| = 3d+6 \equiv -d \pmod{4d+6} \in \langle S \rangle$, and $v_{3d+3} \not\sim v_{-1}$ since $|(3d+3)-(-1)| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$. Furthermore, v_1 is adjacent to neither v_{3d+3} nor v_{3d+5} since $|(3d+3)-1| =$

$3d + 2 \equiv -(d + 4) \pmod{4d + 6} \notin \langle S \rangle$ and $|(3d + 5) - 1| = 3d + 4 \equiv -(d + 2) \pmod{4d + 6} \notin \langle S \rangle$. Hence, H'_x is isomorphic to P_4 , and thus is well-covered with $\beta(H'_x) = 2$.

Finally, we note that $v_{k-1} \sim v_{k+1}$ since $|(k + 1) - (k - 1)| = 2 \in S$, hence H''_x is well-covered with $\beta(H''_x) = 1$. Therefore, $V(H_x)$ is well-covered with $\beta(H_x) = 3$.

Case 5.2.10.4 $x = v_{3d+5}$.

Let $H'_x = H_w[\{v_{k-1}, v_{k+1}\}]$, $H''_x = H_w[\{v_1\}]$ and $H'''_x = H_w[\{v_{3d+4}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ and $V(H'''_x)$ forms a partition of $V(H_x)$.

We claim that no vertex in one of the graphs H'_x , H''_x and H'''_x is adjacent to a vertex in either of the other two graphs. Observe that $v_1 \not\sim v_{3d+4}$ since $|(3d + 4) - 1| = 3d + 3 \equiv -(d + 3) \pmod{4d + 6} \notin \langle S \rangle$. Next, we consider v_{3d+4} . Note that $|(3d + 4) - (k - 1)| = 3d - k + 5$ and $|(3d + 4) - (k + 1)| = 3d - k + 3$. Since $d + 4 \leq k \leq 2d - 1$, it follows that $d + 6 \leq 3d - k + 5 \leq 2d + 1$ and $d + 4 \leq 3d - k + 3 \leq 2d - 1$. Hence, v_{3d+4} is adjacent to neither v_{k-1} nor v_{k+1} . Furthermore, from Case 5.2.10.1, we know that v_1 is adjacent to neither v_{k-1} nor v_{k+1} .

Next, we note that $v_{k-1} \sim v_{k+1}$ since $|(k + 1) - (k - 1)| = 2 \in S$, hence H'_x is well-covered with $\beta(H'_x) = 1$. Furthermore, H''_x and H'''_x are well-covered with $\beta(H''_x) = \beta(H'''_x) = 1$. Therefore, H_x is well-covered with $\beta(H_x) = 3$.

This concludes Case 5.2.10.

Case 5.2.11 $w = v_{2d}$ or $w = v_{-2d}$.

By symmetry, we need only examine $w = v_{2d}$. Note that $V(H_w) = \{v_{-(d+5)}, v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_1, v_{2d-1}, v_{2d+1}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $y_0 = v_{-(d+3)}$. Since $v_{-(d+3)}$

is adjacent to $v_{-(d+3)+j}$ and $v_{-(d+3)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+3)}] = \{v_{-(d+5)}, v_{-(d+3)}, v_{-(d+1)}\}$. We will show for each $y \in N_{H_w}[v_{-(d+3)}]$ that $H_y = H_w \setminus N_{H_w}[y]$ is well-covered with $\beta(H_y) = 3$.

Case 5.2.11.1 $y = v_{-(d+3)}$.

Let $H'_y = H_w[\{v_{-1}, v_1\}]$, $H''_y = H_w[\{v_{2d-1}, v_{2d+1}\}]$ and $H'''_y = H_w[\{v_{-(d+4)}, v_{-(d+2)}\}]$. Note that $V(H'_y)$ together with $V(H''_y)$ and $V(H'''_y)$ forms a partition of $V(H_y)$.

We claim that no vertex in one of the graphs H'_y , H''_y and H'''_y is adjacent to a vertex in either of the other two graphs. First, we note that v_1 is adjacent to neither $v_{-(d+4)}$, $v_{-(d+2)}$, v_{2d-1} nor v_{2d+1} since $|1 - (-(d+4))| = d+5 \notin \langle S \rangle$, $|1 - (-(d+2))| = d+3 \notin \langle S \rangle$, $|(2d-1) - 1| = 2d-2 \notin \langle S \rangle$ and $|(2d+1) - 1| = 2d \notin \langle S \rangle$. Similarly, v_{-1} is adjacent to neither $v_{-(d+4)}$, $v_{-(d+2)}$, v_{2d-1} nor v_{2d+1} since $|(-1) - (-(d+4))| = d+3 \notin \langle S \rangle$, $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$, $|(2d-1) - (-1)| = 2d \notin \langle S \rangle$ and $|(2d+1) - (-1)| = 2d+2 \notin \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{2d-1} nor v_{2d+1} since $|(2d-1) - (-(d+2))| = 3d+1 \equiv -(d+5) \pmod{4d+6} \notin \langle S \rangle$ and $|(2d+1) - (-(d+2))| = 3d+3 \equiv -(d+3) \pmod{4d+6} \notin \langle S \rangle$. Furthermore, $v_{-(d+4)}$ is adjacent to neither v_{2d-1} nor v_{2d+1} since $|(2d-1) - (-(d+4))| = 3d+3 \equiv -(d+3) \pmod{4d+6} \notin \langle S \rangle$ and $|(2d+1) - (-(d+4))| = 3d+5 \equiv -(d+1) \pmod{4d+6} \notin \langle S \rangle$.

To conclude $v_{-1} \sim v_1$ since $|1 - (-1)| = 2 \in S$; $v_{2d-1} \sim v_{2d+1}$ since $|(2d+1) - (2d-1)| = 2 \in S$; and $v_{-(d+4)} \sim v_{-(d+2)}$ since $|(-(d+2) - (-(d+4)))| = 2 \in S$. Hence, H'_y , H''_y and H'''_y are well-covered with $\beta(H'_y) = \beta(H''_y) = \beta(H'''_y) = 1$. Therefore, H_y is well-covered with $\beta(H_y) = 3$.

Case 5.2.11.2 $y = v_{-(d+5)}$.

Let $H'_y = H_w[\{v_{-1}, v_1\}]$, $H''_y = H_w[\{v_{2d-1}\}]$ and $H'''_y = H_w[\{v_{-(d+4)}\}]$. Note that $V(H'_y)$ together with $V(H''_y)$ and $V(H'''_y)$ forms a partition of $V(H_y)$.

From Case 5.2.11.1, we know that $v_{-(d+4)}$ is adjacent to neither v_{-1} , v_1 nor v_{2d-1} ; and v_{2d-1} is adjacent to neither v_{-1} nor v_1 . Hence, no vertex in one of the graphs H'_y , H''_y and H'''_y is adjacent to a vertex in either of the other two graphs.

Next, we note that $v_1 \sim v_{-1}$ since $|1 - (-1)| = 2 \in S$, hence H'_y is well-covered with $\beta(H'_y) = 1$. Furthermore, H''_y and H'''_y are well-covered with $\beta(H''_y) = \beta(H'''_y) = 1$. Therefore, H_y is well-covered with $\beta(H_y) = 3$.

Case 5.2.11.3 $y = v_{-(d+1)}$.

Let $H'_y = H_w[\{v_1\}]$, $H''_y = H_w[\{v_{2d-1}, v_{2d+1}\}]$ and $H'''_y = H_w[\{v_{-(d+2)}\}]$. Note that $V(H'_y)$ together with $V(H''_y)$ and $V(H'''_y)$ forms a partition of $V(H_y)$.

From Case 5.2.11.1, we know that v_1 is adjacent to neither $v_{-(d+2)}$, v_{2d-1} nor v_{2d+1} ; and $v_{-(d+2)}$ is adjacent to neither v_{2d-1} nor v_{2d+1} . Hence, no vertex in one of the graphs H'_y , H''_y and H'''_y is adjacent to a vertex in either of the other two graphs.

Next, we note that $v_{2d-1} \sim v_{2d+1}$ since $|(2d+1) - (2d-1)| = 2 \in S$, hence H''_y is well-covered with $\beta(H''_y) = 1$. Furthermore, H'_y and H'''_y are well-covered with $\beta(H'_y) = \beta(H'''_y) = 1$. Therefore, H_y is well-covered with $\beta(H_y) = 3$.

This concludes Case 5.2.11.

Case 5.2.12 $w = v_{2d+1}$ or $w = v_{-(2d+1)}$.

By symmetry, we need only examine $w = v_{2d+1}$. Note that $V(H_w) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_1, v_{2d}, v_{2d+2}\}$. To show that H_w is well-covered we

are going to apply Lemma 5.1 with $z_0 = v_{-(d+1)}$. Since $v_{-(d+1)}$ is adjacent to $v_{-(d+1)+j}$ and $v_{-(d+1)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+1)}] = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+1)}, v_{-1}\}$. We will show for each $z \in N_{H_w}[v_{-(d+1)}]$ that $H_z = H_w \setminus N_{H_w}[z]$ is well-covered with $\beta(H_z) = 3$.

Case 5.2.12.1 $z = v_{-(d+1)}$.

Let $H'_z = H_w[\{v_{2d}, v_{2d+2}\}]$, $H''_z = H_w[\{v_1\}]$ and $H'''_z = H_w[\{v_{-(d+2)}\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ and $V(H'''_z)$ forms a partition of $V(H_z)$.

We claim that no vertex in one of the graphs H'_z , H''_z and H'''_z is adjacent to a vertex in either of the other two graphs. Observe that v_1 is adjacent to neither v_{2d} nor v_{2d+2} since $|2d-1| = 2d-1 \notin \langle S \rangle$ and $|(2d+2)-1| = 2d+1 \notin \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{2d} nor v_{2d+2} since $|(2d) - (-(d+2))| = 3d+2 \equiv -(d+4) \pmod{4d+6} \notin \langle S \rangle$ and $|(2d+2) - (-(d+2))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$. Furthermore, $v_1 \not\sim v_{-(d+2)}$ since $|1 - (-(d+2))| = d+3 \notin \langle S \rangle$.

Next, we note that $v_{2d+2} \sim v_{2d}$ since $|(2d+2) - 2d| = 2 \in S$, hence H'_z is well-covered with $\beta(H'_z) = 1$. Furthermore, H''_z and H'''_z are well-covered with $\beta(H''_z) = \beta(H'''_z) = 1$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

Case 5.2.12.2 $z = v_{-1}$.

Let $H'_z = H_w[\{v_{-(d+4)}, v_{-(d+2)}, v_{2d}, v_{2d+2}\}]$ and $H''_z = H_w[\{v_{-(d+3)}\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ forms a partition of $V(H_z)$.

We claim that no vertex in H'_z is adjacent to a vertex in H''_z . Observe that $v_{-(d+3)}$ is adjacent to neither $v_{-(d+4)}$ nor $v_{-(d+2)}$ since $|-(d+3) - (-(d+4))| = 1 \notin S$ and $|-(d+2) - (-(d+3))| = 1 \notin S$. Furthermore, $v_{-(d+3)}$ is adjacent to neither

v_{2d} nor v_{2d+2} since $|(2d) - (-(d+3))| = 3d+3 \equiv -(d+3) \pmod{4d+6} \notin \langle S \rangle$ and $|(2d+2) - (-(d+3))| = 3d+5 \equiv -(d+1) \pmod{4d+6} \notin \langle S \rangle$.

Next, we show that H'_z is isomorphic to P_4 . Observe that $v_{2d+2} \sim v_{-(d+4)}$ since $|(2d+2) - (-(d+4))| = 3d+6 \equiv -d \pmod{4d+6} \in \langle S \rangle$, and $v_{2d} \not\sim v_{-(d+4)}$ since $|(2d) - (-(d+4))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{2d} nor v_{2d+2} since $|(2d+2) - (-(d+2))| = 3d+4 \equiv -(d+2) \pmod{4d+6} \notin \langle S \rangle$ and $|(2d) - (-(d+2))| = 3d+2 \equiv -(d+4) \pmod{4d+6} \notin \langle S \rangle$. Furthermore, $v_{2d+2} \sim v_{2d}$ since $|(2d+2) - (2d)| = 2 \in S$, and $v_{-(d+2)} \sim v_{-(d+4)}$ since $|-(d+2) - (-(d+4))| = 2 \in S$. Hence, H'_z is isomorphic to P_4 , and thus is well-covered with $\beta(H'_z) = 2$.

Furthermore, H''_z is well-covered with $\beta(H''_z) = 1$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

Case 5.2.12.3 $z = v_{-(d+3)}$.

Let $H'_z = H_w[\{v_{-(d+4)}, v_{-(d+2)}, v_{2d}, v_{2d+2}\}]$ and $H''_z = H_w[\{v_{-1}, v_1\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ forms a partition of $V(H_z)$.

We claim that no vertex in H'_z is adjacent to a vertex in H''_z . Observe that v_1 is adjacent to neither $v_{-(d+4)}$, $v_{-(d+2)}$, v_{2d} nor v_{2d+2} since $|1 - (-(d+4))| = d+5 \notin \langle S \rangle$, $|1 - (-(d+2))| = d+3 \notin \langle S \rangle$, $|2d-1| = 2d-1 \notin \langle S \rangle$ and $|(2d+2) - 1| = 2d+1 \notin \langle S \rangle$. Similarly, v_{-1} is adjacent to neither $v_{-(d+4)}$, $v_{-(d+2)}$, v_{2d} nor v_{2d+2} since $|(-1) - (-(d+4))| = d+3 \notin \langle S \rangle$, $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$, $|2d - (-1)| = 2d+1 \notin \langle S \rangle$ and $|(2d+2) - (-1)| = 2d+3 \notin \langle S \rangle$.

We now note that $v_1 \sim v_{-1}$ since $|1 - (-1)| = 2 \in S$, hence H''_z is well-covered with $\beta(H''_z) = 1$. Furthermore, from Case 5.2.12.2, we know that H'_z is well-covered with $\beta(H'_z) = 2$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

Case 5.2.12.4 $z = v_{-(d+4)}$.

Let $H'_z = H_w[\{v_{-1}, v_1\}]$, $H''_z = H_w[\{v_{2d}\}]$ and $H'''_z = H_w[\{v_{-(d+3)}\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ and $V(H'''_z)$ forms a partition of $V(H_z)$.

We claim that no vertex in one of the graphs H'_z , H''_z and H'''_z is adjacent to a vertex in either of the other two graphs. Observe that v_1 is adjacent to neither $v_{-(d+3)}$ nor v_{2d} since $|1 - (-(d+3))| = d+4 \notin \langle S \rangle$ and $|2d - 1| = 2d - 1 \notin \langle S \rangle$. Next, we note that v_{-1} is adjacent to neither $v_{-(d+3)}$ nor v_{2d} since $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$ and since $|2d - (-1)| = 2d+1 \notin \langle S \rangle$. Furthermore, $v_{2d} \not\sim v_{-(d+3)}$ since $|(2d) - (-(d+3))| = 3d+3 \equiv -(d+3) \pmod{4d+6} \notin \langle S \rangle$.

Next, we note that $v_1 \sim v_{-1}$ since $|1 - (-1)| = 2 \in S$, hence H'_z is well-covered with $\beta(H'_z) = 1$. Furthermore, H''_z and H'''_z are well-covered with $\beta(H''_z) = \beta(H'''_z) = 1$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

This concludes Case 5.2.12.

Hence, G is well-covered with $\beta(G) = 6$, concluding the proof of Case (v).

We now proceed to prove the ‘only if’ direction.

Case 5.2.13 $n = 3d + 3$.

Then $V(H) = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{-1}\} \cup \{v_1\} \cup \{v_i: d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. Let $I_1 = \{v_0, v_{d+1}, v_{-(d+1)}\}$. Observe that v_0 is adjacent to neither v_{d+1} nor $v_{-(d+1)}$ since $|(d+1) - 0| = d+1 \notin \langle S \rangle$ and $|0 - (-(d+1))| = d+1 \notin \langle S \rangle$. We also note that $v_{d+1} \not\sim v_{-(d+1)}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+1) \pmod{3d+3} \notin \langle S \rangle$. Hence, I_1 is an independent set in G . Since v_{d+1} is adjacent to v_{d+1+j} and v_{d+1-j} for each j in the set S , $N_H[v_{d+1}] = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+2)\} \cup \{v_1\} \cup \{v_i: d+3 \leq$

$i \leq \lfloor \frac{n}{2} \rfloor \}$. Furthermore, $v_{d+2} \sim v_{-(d+1)}$ since $|(d+2) - (-(d+1))| = 2d+3 \equiv -d \pmod{3d+3} \in \langle S \rangle$. Hence, I_1 is a maximal independent set in G .

Now, let $I_2 = \{v_{-(d+1)}, v_{-d}, v_d, v_{d+1}\}$. Observe that v_d is adjacent to neither v_{d+1} nor v_{-d} since $|(d+1) - d| = 1 \notin S$ and $|d - (-d)| = 2d \equiv -(d+3) \pmod{3d+3} \notin \langle S \rangle$. Next, we note that $v_{-(d+1)}$ is adjacent to neither v_d nor v_{d+1} since $|d - (-(d+1))| = 2d+1 \equiv -(d+2) \pmod{3d+3} \notin \langle S \rangle$ and $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+1) \pmod{3d+3} \notin \langle S \rangle$. By symmetry, we can also deduce that v_{-d} is adjacent to neither $v_{-(d+1)}$ nor v_{d+1} . Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 5.2.14 $3d+5 \leq n \leq 4d+4$.

Let $I' = \{v_{-(d+2)}, v_{-(d+1)}, v_{d+1}\}$. Observe that $v_{-(d+2)} \not\sim v_{-(d+1)}$ since $|-(d+1) - (-(d+2))| = 1 \notin S$. Next, we consider v_{d+1} . Note that $|(d+1) - (-(d+2))| = 2d+3$ and $|(d+1) - (-(d+1))| = 2d+2$. Given our assumption that $3d+5 \leq n \leq 4d+4$, it follows that $d+2 \leq n - (2d+3) \leq 2d+1$ and $d+3 \leq n - (2d+2) \leq 2d+2$. Therefore, v_{d+1} is adjacent to neither $v_{-(d+2)}$ nor $v_{-(d+1)}$. Hence, I' is an independent set in G .

Now, let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_0, v_d, v_{d+2}\}$. First, let $K_1 = \{v_d\}$. Note that v_d is adjacent to both v_0 and v_{d+2} since $|d - 0| = d \in S$ and $|(d+2) - d| = 2 \in S$. Thus, K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_0, v_{d+2}\}$. Observe that $v_{d+2} \not\sim v_0$ since $|(d+2) - 0| = d+2 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.2.15 $n = 4d+5$.

Let $I' = \{v_{-(2d+2)}, v_{-(2d+1)}, v_0\}$. Observe that v_0 is adjacent to neither $v_{-(2d+2)}$ nor $v_{-(2d+1)}$ since $|0 - (-(2d+2))| = 2d+2 \notin \langle S \rangle$ and $|0 - (-(2d+1))| = 2d+1 \notin \langle S \rangle$.

Furthermore, $v_{-(2d+2)} \not\sim v_{-(2d+1)}$ since $|-(2d+1) - (-(2d+2))| = 1 \notin S$. Hence, I' is an independent set in G .

Now, let H_1 be the component of $G \setminus N[I']$ containing v_1 . It follows that $V(H_1) = \{v_{-1}, v_1, v_{d+1}\}$. First, let $K_1 = \{v_1\}$. Note that v_1 is adjacent to both v_{-1} and v_{d+1} since $|1 - (-1)| = 2 \in S$ and $|(d+1) - 1| = d \in S$. Thus, K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_{d+1}\}$. Observe that $v_{d+1} \not\sim v_{-1}$ since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.2.16 $n \geq 4d + 7$.

Let $I' = \{v_{-(d+3)}, v_0, v_{2d+2}, v_{2d+3}\}$. Observe that $v_{-(d+3)} \not\sim v_0$ since $|0 - (-(d+3))| = d+3 \notin \langle S \rangle$, and $v_{2d+2} \not\sim v_{2d+3}$ since $|(2d+3) - (2d+2)| = 1 \notin S$. Next, we consider $v_{-(d+3)}$. Note that $|(2d+2) - (-(d+3))| = 3d+5$ and $|(2d+3) - (-(d+3))| = 3d+6$. Given our assumption that $n \geq 4d + 7$, it follows that $n - (3d+5) \geq d+2$ and $n - (3d+6) \geq d+1$. Therefore, $v_{-(d+3)}$ is adjacent to neither v_{2d+2} nor v_{2d+3} . Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_1 . It follows that $V(H_1) = \{v_{-1}, v_1, v_{d+1}\}$. First, let $K_1 = \{v_1\}$. Note that v_1 is adjacent to both v_{-1} and v_{d+1} since $|1 - (-1)| = 2 \in S$ and $|(d+1) - 1| = d \in S$. Thus, K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_{d+1}\}$. Observe that $v_{d+1} \not\sim v_{-1}$ since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

■

A characterization of the well-covered graphs in Class 8 can now be stated.

Theorem 5.3 *Let n and d be integers with $3 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{1\} \cup \{3, 4, \dots, d\})$ is well-covered if and only if one of the following conditions holds:*

(i) $d \geq 5$ and $2d + 7 \leq n \leq 3d + 2$, or

(ii) $d \geq 4$ and $2d \leq n \leq 2d + 3$, or

(iii) $d \geq 4$ and $n = 2d + 5$, or

(iv) G is one of the following: $C(6, \{1, 3\})$, $C(7, \{1, 3\})$, $C(8, \{1, 3\})$, $C(9, \{1, 3\})$, $C(11, \{1, 3\})$ or $C(13, \{1, 3\})$.

Furthermore, if $2d + 7 \leq n \leq 3d + 2$ or if G is one of $C(8, \{1, 3\})$ or $C(11, \{1, 3\})$ then $\beta(G) = 4$; if $2d \leq n \leq 2d + 3$ or $G = C(7, \{1, 3\})$ then $\beta(G) = 2$; if $n = 2d + 5$ or if G is one of $C(6, \{1, 3\})$ or $C(9, \{1, 3\})$ then $\beta(G) = 3$; and if $G = C(13, \{1, 3\})$ then $\beta(G) = 5$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. In each case let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Note that $V(H) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{-2}\} \cup \{v_2\} \cup \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. By Corollary 2.6, it suffices to show that in Case (i) H is well-covered with $\beta(H) = 3$; in Case (ii) H is well-covered with $\beta(H) = 1$; and in Case (iii) H is well-covered with $\beta(H) = 2$.

(i) $d \geq 5$ and $2d + 7 \leq n \leq 3d + 2$.

We must consider the following four cases:

Case 5.3.1 $d \geq 5$ and $n = 2d + 7$.

In this case $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_2, v_{d+1}, v_{d+2}, v_{d+3}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_2$. Since v_2 is adjacent to v_{2+j} and v_{2-j} for each j in the set S , it follows that $N_H[v_2] = \{v_{-2}, v_2, v_{d+1}, v_{d+2}\}$. We will show for each $w \in N_H[v_2]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.3.1.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_2$. We first note that $V(H_w) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+3}\}$. Observe that v_{d+3} is adjacent to both $v_{-(d+3)}$ and $v_{-(d+1)}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -1 \pmod{2d+7} \in \langle S \rangle$ and $|(d+3) - (-(d+1))| = 2d+4 \equiv -3 \pmod{2d+7} \in \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to both $v_{-(d+3)}$ and $v_{-(d+1)}$ since $|-(d+2) - (-(d+3))| = 1 \in S$ and $|-(d+1) - (-(d+2))| = 1 \in S$. Furthermore, $v_{d+3} \not\sim v_{-(d+2)}$ since $|(d+3) - (-(d+2))| = 2d+5 \equiv -2 \pmod{2d+7} \notin \langle S \rangle$, and $v_{-(d+3)} \not\sim v_{-(d+1)}$ since $|-(d+1) - (-(d+3))| = 2 \notin S$. Hence, H_w is isomorphic to C_4 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.3.1.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{d+3}\}$. Observe that $v_{d+3} \not\sim v_{-2}$ since $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.3.1.3 $w = v_{d+2}$.

In this case $V(H_w) = \{v_{-(d+3)}, v_{-2}\}$. Observe that $v_{-(d+3)} \not\sim v_{-2}$ since $|(-2) - (-(d+3))| = d+1 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.3.1.

Case 5.3.2 $d \geq 6$ and $n = 2d + 8$.

In this case $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_2, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_2$. Since v_2 is adjacent to v_{2+j} and v_{2-j} for each j in the set S , it follows that $N_H[v_2] = \{v_{-2}, v_2, v_{d+1}, v_{d+2}\}$. We will show for each $w \in N_H[v_2]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.3.2.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_2$. We first note that $V(H_w) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+3}, v_{d+4}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{-(d+3)}$. Since $v_{-(d+3)}$ is adjacent to $v_{-(d+3)+j}$ and $v_{-(d+3)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+3)}] = \{v_{-(d+3)}, v_{-(d+2)}, v_{d+4}\}$. We will show for each $u \in N_{H_w}[v_{-(d+3)}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.3.2.1.1 $u = v_{-(d+3)}$

In this case $V(H_u) = \{v_{-(d+1)}, v_{d+3}\}$. Observe that $v_{d+3} \sim v_{-(d+1)}$ since $|(d+3) - (-(d+1))| = 2d+4 \equiv -4 \pmod{2d+8} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.2.1.2 $u = v_{-(d+2)}$.

In this case $V(H_u) = \{v_{d+4}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.2.1.3 $u = v_{d+4}$.

In this case $V(H_u) = \{v_{-(d+2)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.3.2.1.

Case 5.3.2.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{d+3}\}$. Observe that $v_{d+3} \not\sim v_{-2}$ since $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.3.2.3 $w = v_{d+2}$.

In this case $V(H_w) = \{v_{-2}, v_{d+4}\}$. Observe that $v_{d+4} \not\sim v_{-2}$ since $|(d+4) - (-2)| = d+6 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.3.2.

Case 5.3.3 $d \geq 7$ and $n = 2d + 9$.

Note that $V(H) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_2, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_2$.

Since v_2 is adjacent to v_{2+j} and v_{2-j} for each j in the set S , it follows that

$N_H[v_2] = \{v_{-2}, v_2, v_{d+1}, v_{d+2}\}$. We will show for each $w \in N_H[v_2]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.3.3.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_2$. We first note that $V(H_w) =$

$\{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+3}, v_{d+4}\}$. To show that H_w is well-covered

we are going to apply Lemma 5.1 with $u_0 = v_{-(d+3)}$. Since $v_{-(d+3)}$ is adjacent

to $v_{-(d+3)+j}$ and $v_{-(d+3)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+3)}] =$

$\{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{d+3}\}$. We will show for each $u \in N_{H_w}[v_{-(d+3)}]$ that

$H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.3.3.1.1 $u = v_{-(d+3)}$.

In this case $V(H_u) = \{v_{-(d+1)}, v_{d+4}\}$. Observe that $v_{d+4} \sim v_{-(d+1)}$ since $|(d+4) - (-(d+1))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.3.1.2 $u = v_{-(d+4)}$.

In this case $V(H_u) = \{v_{-(d+2)}, v_{d+3}\}$. Observe that $v_{d+3} \sim v_{-(d+2)}$ since $|(d+3) - (-(d+2))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.3.1.3 $u = v_{-(d+2)}$.

In this case $V(H_u) = \{v_{-(d+4)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.3.1.4 $u = v_{d+3}$.

In this case $V(H_u) = \{v_{-(d+4)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.3.3.1.

Case 5.3.3.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{d+3}\}$. Observe that $v_{d+3} \not\sim v_{-2}$ since $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.3.3.3 $w = v_{d+2}$.

In this case $V(H_w) = \{v_{-2}, v_{d+4}\}$. Observe that $v_{d+4} \not\sim v_{-2}$ since $|(d+4) - (-2)| = d+6 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.3.3.

Case 5.3.4 $d \geq 8$ and $2d + 10 \leq n \leq 3d + 2$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{-2}$. Since v_{-2} is adjacent to v_{-2+j} and v_{-2-j} for each j in the set S , it follows that $N_H[v_{-2}] = \{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_2\}$. We will show for each $w \in N_H[v_{-2}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.3.4.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_{-2}$. We first note that $V(H_w) = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+3)\} \cup \{v_i: d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{d+1}$. Since v_{d+1} is adjacent to v_{d+1+j} and v_{d+1-j} for each j in the set S , it follows that $N_{H_w}[v_{d+1}] = \{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+3)\} \cup \{v_{d+1}, v_{d+2}\} \cup \{v_i: d+4 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We will show for each $u \in N_{H_w}[v_{d+1}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.3.4.1.1 $u = v_{d+1}$.

In this case $V(H_u) = \{v_{d+3}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.4.1.2 $u = v_{d+2}$.

In this case $V(H_u) = \{v_{d+4}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.4.1.3 $u = v_k$ or $u = v_{-k}$ for $d+4 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By symmetry, we need only examine $u = v_k$. Since 2 is not in S , we can deduce that v_k is adjacent to neither v_{k-2} nor v_{k+2} , and thence $V(H_u) = \{v_{k-2}, v_{k+2}\}$. Note that $v_{k+2} \sim v_{k-2}$ since $|(k+2) - (k-2)| = 4 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.3.4.1.4 $u = v_{-(d+3)}$.

In this case $V(H_u) = \{v_{-(d+5)}\}$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.3.4.1.

Case 5.3.4.2 $w = v_{-(d+2)}$.

In this case $V(H_w) = \{v_2, v_{-(d+4)}\}$. Observe that $v_{-(d+4)} \not\sim v_2$ since $|2 - (-(d+4))| = d+6 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.3.4.3 $w = v_{-(d+1)}$.

In this case $V(H_w) = \{v_2, v_{-(d+3)}\}$. Observe that $v_{-(d+3)} \not\sim v_2$ since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.3.4.

(ii) $d \geq 4$ and $2d \leq n \leq 2d+3$.

We must consider the following three cases:

Case 5.3.5 $n = 2d + k$ for $k = 0$ or 1 .

In this case $V(H) = \{v_{-2}, v_2\}$. Observe that $v_2 \sim v_{-2}$ since $|2 - (-2)| = 4 \in S$. H is therefore a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$.

Case 5.3.6 $n = 2d + 2$.

In this case $V(H) = \{v_{-2}, v_2, v_{d+1}\}$. Observe that $v_2 \sim v_{d+1}$ since $|(d+1) - 2| = d-1 \in S$. Next, we note that $v_{-2} \sim v_{d+1}$ since $|(d+1) - (-2)| = d+3 \equiv -(d-1) \pmod{2d+2} \in \langle S \rangle$. Furthermore, $v_2 \sim v_{-2}$. H is therefore a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$.

Case 5.3.7 $n = 2d + 3$.

In this case $V(H) = \{v_{-(d+1)}, v_{-2}, v_2, v_{d+1}\}$. Observe that $v_{-2} \sim v_{-(d+1)}$ since $|(-2) - (-(d+1))| = d - 1 \in S$. Next, we note that v_{d+1} is adjacent to both $v_{-(d+1)}$ and v_{-2} since $|(d+1) - (-(d+1))| = 2d + 2 \equiv -1 \pmod{2d+3} \in \langle S \rangle$ and $|(d+1) - (-2)| = d + 3 \equiv -d \pmod{2d+3} \in \langle S \rangle$. By symmetry, we can also deduce that v_2 is adjacent to both $v_{-(d+1)}$ and v_{d+1} . Furthermore, $v_2 \sim v_{-2}$. H is therefore a complete graph and $|I \cap V(H)| = 1$. Hence, $|I| = 2$, G is well-covered and $\beta(G) = 2$.

(iii) $d \geq 4$ and $n = 2d + 5$.

In this case $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_2, v_{d+1}, v_{d+2}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_2$. Since v_2 is adjacent to v_{2+j} and v_{2-j} for each j in the set S , it follows that $N_H[v_2] = \{v_{-2}, v_2, v_{d+1}, v_{d+2}\}$. We will show for each $w \in N_H[v_2]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 1$.

Case 5.3.8 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_2$. We first note that $V(H_w) = \{v_{-(d+2)}, v_{-(d+1)}\}$. Observe that $v_{-(d+2)} \sim v_{-(d+1)}$ since $|-(d+1) - (-(d+2))| = 1 \in S$. Hence, H_w is well-covered with $\beta(H_w) = 1$.

Case 5.3.9 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{-(d+2)}\}$. Observe that $v_{-(d+2)} \sim v_{-2}$ since $|(-2) - (-(d+2))| = d \in S$. Hence, H_w is well-covered with $\beta(H_w) = 1$.

Case 5.3.10 $w = v_{d+2}$.

In this case $V(H_w) = \{v_{-(d+1)}, v_{-2}\}$. Observe that $v_{-(d+1)} \sim v_{-2}$ since $|(-2) - (-(d+1))| = d-1 \in S$. Hence, H_w is well-covered with $\beta(H_w) = 1$.

Hence, G is well-covered and $\beta(G) = 3$.

(iv) G is one of the following $C(6, \{1, 3\})$, $C(7, \{1, 3\})$, $C(8, \{1, 3\})$, $C(9, \{1, 3\})$, $C(11, \{1, 3\})$ or $C(13, \{1, 3\})$.

Note that $C(6, \{1, 3\})$ and $C(9, \{1, 3\})$ are well-covered and $\beta(G) = 3$; $C(7, \{1, 3\})$ is well-covered and $\beta(G) = 2$; $C(8, \{1, 3\})$ and $C(11, \{1, 3\})$ are well-covered and $\beta(G) = 4$; and $C(13, \{1, 3\})$ is well-covered and $\beta(G) = 5$.

We now proceed to prove the ‘only if’ direction.

Case 5.3.11 $d \geq 3$ and $n = 2d + 4$.

First, we note that $C(10, \{1, 3\})$ is not well-covered.

Next, we consider the case where $d \geq 4$. Let $I' = \{v_{d+2}\}$ and let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-1}, v_0, v_1, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Observe that $v_{-1} \not\sim v_1$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.3.12 $d \geq 3$ and $n = 2d + 6$.

First, we note that $C(12, \{1, 3\})$ is not well-covered.

Next, we consider the case where $d \geq 4$. Let $I' = \{v_{-(d+2)}, v_{d+2}\}$. Observe that $v_{-(d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -2 \pmod{2d+6} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-1}, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Observe that $v_{-1} \not\sim v_1$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.3.13 $d = 3$ and $n \geq 3d + 5$.

Let $I' = \{v_{-(d+2)}, v_{d+2}\}$. First, we show that $v_{-(d+2)} \not\sim v_{d+2}$. Note that $|(d + 2) - (-(d + 2))| = 2d + 4$. Given our assumption that $n \geq 3d + 5$, it follows that $n - (2d + 4) \geq d + 1$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_{-1}, v_0, v_1, v_3\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_{-1}, v_1, v_3\}$. Observe that v_1 is adjacent to neither v_{-1} nor v_3 since $|1 - (-1)| = 2 \notin S$ and $|3 - 1| = 2 \notin S$. Next, we note that v_3 is adjacent to neither v_{-3} nor v_{-1} since $|3 - (-3)| = 6 \notin S$ and $|3 - (-1)| = 4 \notin S$. By symmetry, we can also deduce that v_{-3} is adjacent to neither v_{-1} nor v_1 . Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.3.14 $d \geq 4$ and $n \geq 3d + 3$.

Case 5.3.14.1 $n = 3d + 3$.

Let $I' = \{v_{-(d+1)}, v_{d+1}\}$. Observe that $v_{-(d+1)} \not\sim v_{d+1}$ since $|(d+1) - (-(d+1))| = 2d + 2 \equiv -(d+1) \pmod{3d+3} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_0, v_{d-1}\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-(d-1)}, v_{d-1}\}$. Observe that $v_{-(d-1)} \not\sim v_{d-1}$ since $|(d-1) - (-(d-1))| = 2d - 2 \equiv -(d+5) \pmod{3d+3} \notin \langle S \rangle$. Therefore, K_2 is an independent

set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.3.14.2 $n = 3d + 4$.

Let $I' = \{v_{-(d+1)}, v_{d+2}\}$. Observe that $v_{-(d+1)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+1))| = 2d + 3 \equiv -(d+1) \pmod{3d+4} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_0, v_1, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-(d-1)}, v_d\}$. Observe that $v_{-(d-1)} \not\sim v_d$ since $|d - (-(d-1))| = 2d - 1 \equiv -(d+5) \pmod{3d+4} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.3.14.3 $n = 3d + 5$.

Let $I' = \{v_{-(d+2)}, v_{d+2}\}$. Observe that $v_{-(d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+2))| = 2d + 4 \equiv -(d+1) \pmod{3d+5} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-1}, v_0, v_1, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Observe that $v_{-1} \not\sim v_1$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 6$ is also not well-covered.

Case 5.3.14.4 $n = 3d + 7$.

Let $I' = \{v_{-(d+2)}, v_{d+2}, v_{d+4}\}$. Observe that $v_{d+2} \not\sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \notin S$. Next, we note that $v_{d+4} \not\sim v_{-(d+2)}$ since $|(d+4) - (-(d+2))| = 2d+6 \equiv -(d+1) \pmod{3d+7} \notin \langle S \rangle$. Furthermore, $v_{d+2} \not\sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+3) \pmod{3d+7} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-1}, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Observe that $v_{-1} \not\sim v_1$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 8$ is also not well-covered.

Case 5.3.14.5 $n \geq 3d + 9$.

Let $I' = \{v_{-(d+4)}, v_{-(d+2)}, v_{d+2}, v_{d+4}\}$. Observe that $v_{d+2} \not\sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \notin S$. Next, we consider $v_{-(d+2)}$. Note that $|(d+2) - (-(d+2))| = 2d+4$ and $|(d+4) - (-(d+2))| = 2d+6$. Given our assumption that $n \geq 3d+9$, it follows that $n - (2d+4) \geq d+5$ and $n - (2d+6) \geq d+3$. Hence, $v_{-(d+2)}$ is adjacent to neither v_{d+2} nor v_{d+4} . Finally, we show that $v_{d+4} \not\sim v_{-(d+4)}$. Note that $|(d+4) - (-(d+4))| = 2d+8$. Given our assumption that $n \geq 3d+9$, it follows that $n - (2d+8) \geq d+1$. By symmetry, we can also deduce that $v_{-(d+4)}$ is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} . Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-1}, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Observe that $v_{-1} \not\sim v_1$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. ■

A characterization of the well-covered graphs in Class 9 can now be stated.

Theorem 5.4 *Let n and d be integers with $4 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{1\} \cup \{4, 5, \dots, d\})$ is well-covered if and only if one of the following conditions holds:*

- (i) $d \geq 6$ and $2d \leq n \leq 2d + 3$, or

(ii) $d \geq 6$ and $n = 2d + 5$, or

(iii) $d \geq 6$ and $2d + 8 \leq n \leq 3d + 2$, or

(iv) G is one of the following: $C(10, \{1, 4, 5\})$, $C(11, \{1, 4, 5\})$, $C(8, \{1, 4\})$,
 $C(10, \{1, 4\})$, $C(11, \{1, 4\})$, $C(12, \{1, 4\})$, or $C(13, \{1, 4\})$.

Furthermore, if $2d \leq n \leq 2d+3$ or if G is one of $C(10, \{1, 4, 5\})$ or $C(11, \{1, 4, 5\})$ then $\beta(G) = 2$; if $n = 2d+5$ or $G = C(8, \{1, 4\})$ then $\beta(G) = 3$; if $2d+8 \leq n \leq 3d+2$ or if G is one of $C(10, \{1, 4\})$, $C(11, \{1, 4\})$ or $C(12, \{1, 4\})$ then $\beta(G) = 4$; and if $G = C(13, \{1, 4\})$ then $\beta(G) = 5$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n-1\}$. First, we prove the ‘if’ direction. In each case let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Note that $V(H) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{-3}, v_{-2}\} \cup \{v_2, v_3\} \cup \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. By Corollary 2.6, it suffices to show that in Case (i) H is well-covered with $\beta(H) = 1$; in Case (ii) H is well-covered with $\beta(H) = 2$; and in Case (iii) H is well-covered with $\beta(H) = 3$.

(i) $d \geq 6$ and $2d \leq n \leq 2d + 3$.

We must consider the following three cases:

Case 5.4.1 $n = 2d + k$ for $k = 0$ or 1 .

In this case $V(H) = \{v_{-3}, v_{-2}, v_2, v_3\}$. We claim that $V(H)$ is a complete graph. Observe that v_2 is adjacent to v_{-3} , v_3 and v_{-2} since $|2 - (-3)| = 5 \in S$, $|3 - 2| = 1 \in S$ and $|2 - (-2)| = 4 \in S$. Next, we note that $v_3 \sim v_{-3}$ since $|3 - (-3)| = 6 \in S$. By symmetry, we can also deduce that v_{-2} is adjacent to both v_{-3} and v_3 . Hence, H is a complete graph and $\beta(H) = 1$. Therefore, G is well-covered and $\beta(G) = 2$.

Case 5.4.2 $n = 2d + 2$.

In this case $V(H) = \{v_{-3}, v_{-2}, v_2, v_3, v_{d+1}\}$. We claim that $V(H)$ is a complete graph. Observe that v_{d+1} is adjacent to both v_2 and v_3 since $|(d+1) - 2| = d - 1 \in S$ and $|(d+1) - 3| = d - 2 \in S$. Next, we note that v_{d+1} is adjacent to both v_{-3} and v_{-2} since $|(d+1) - (-2)| = d + 3 \equiv -(d-1) \pmod{2d+2} \in \langle S \rangle$ and $|(d+1) - (-3)| = d + 4 \equiv -(d-2) \pmod{2d+2} \in \langle S \rangle$. Furthermore, from Case 5.4.1, we know that v_2 is adjacent to v_{-3} , v_{-2} and v_3 ; v_3 is adjacent to both v_{-3} and v_{-2} ; and $v_{-2} \sim v_{-3}$. Hence, H is a complete graph and $\beta(H) = 1$. Therefore, G is well-covered and $\beta(G) = 2$.

Case 5.4.3 $n = 2d + 3$.

In this case $V(H) = \{v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}\}$. We claim that $V(H)$ is a complete graph. Observe that v_{d+2} is adjacent to v_2 , v_3 and v_{d+1} since $|(d+2) - 2| = d \in S$, $|(d+2) - 3| = d - 1 \in S$ and $|(d+2) - (d+1)| = 1 \in S$. Next, we note that v_{d+2} is adjacent to both v_{-3} and v_{-2} since $|(d+2) - (-2)| = d + 4 \equiv -(d-1) \pmod{2d+3} \in \langle S \rangle$ and $|(d+2) - (-3)| = d + 5 \equiv -(d-2) \pmod{2d+3} \in \langle S \rangle$. Furthermore, from Case 5.4.1 and 5.4.2, we know that v_2 is adjacent to v_{-3} , v_{-2} and v_3 ; v_3 is adjacent to both v_{-3} and v_{-2} ; v_{d+1} is adjacent to v_{-3} , v_{-2} , v_2 and v_3 ; and $v_{-2} \sim v_{-3}$. Hence, H is a complete graph and $\beta(H) = 1$. Therefore, G is well-covered and $\beta(G) = 2$.

(ii) $d \geq 6$ and $n = 2d + 5$.

Let $H_1 = G[\{v_2, v_3, v_{d+1}, v_{d+2}\}]$ and $H_2 = G[\{v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}\}]$. Note that $V(H_1)$ together with $V(H_2)$ forms a partition of $V(H)$.

We claim that H_i is a complete graph for $i = 1$ and 2 . Observe that v_2 is adjacent to v_3 , v_{d+1} and v_{d+2} since $|3 - 2| = 1 \in S$, $|(d+1) - 2| = d - 1 \in S$

and $|(d+2) - 2| = d \in S$. Next, we note that v_3 is adjacent to both v_{d+1} and v_{d+2} since $|(d+1) - 3| = d - 2 \in S$ and $|(d+2) - 3| = d - 1 \in S$. Furthermore, $v_{d+1} \sim v_{d+2}$ since $|(d+2) - (d+1)| = 1 \in S$. Hence, H_1 is a complete graph. Similarly, H_2 is a complete graph.

Note that $V(H) \neq \emptyset$, hence $|I \cap V(H)| \neq \emptyset$. Without loss of generality, let $v_k \in I \cap V(H_1)$. Next, consider $v_{-(d+1)}$ a vertex in $V(H_2)$. We claim that $|k - (-(d+1))| > d$. First, we note that $v_{-(d+1)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+1))| = d + 3 \notin \langle S \rangle$ and $|3 - (-(d+1))| = d + 4 \notin \langle S \rangle$. Next, observe that $v_{-(d+1)}$ is adjacent to neither v_{d+1} nor v_{d+2} since $|(d+1) - (-(d+1))| = 2d + 2 \equiv -3 \pmod{2d+5} \notin \langle S \rangle$ and $|(d+2) - (-(d+1))| = 2d + 3 \equiv -2 \pmod{2d+5} \in \langle S \rangle$. Therefore, $v_{-(d+1)} \not\sim v_k$, hence $|I \cap V(H_2)| \neq \emptyset$ and $|I| \geq 3$. Since H_i is complete, it follows that $|I \cap V(H_i)| = 1$ for $i = 1$ and 2 . Hence, $|I| = 3$, G is well-covered and $\beta(G) = 3$.

(iii) $d \geq 6$ and $2d + 8 \leq n \leq 3d + 2$.

We must consider the following three cases:

Case 5.4.4 $d \geq 6$ and $n = 2d + 8$.

Note that $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{-2}$.

Since v_{-2} is adjacent to v_{-2+j} and v_{-2-j} for each j in the set S , it follows that

$N_H[v_{-2}] = \{v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3\}$. We will show for each $w \in N_H[v_{-2}]$

that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.4.4.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_{-2}$. We first note that $V(H_w) =$

$\{v_{-(d+3)}, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. We claim that H_w is isomorphic to C_5 . Observe

that v_{d+2} is adjacent to both v_{d+1} and v_{d+3} since $|(d+2) - (d+1)| = 1 \in S$ and $|(d+3) - (d+2)| = 1 \in S$. Similarly, $v_{-(d+3)}$ is adjacent to both v_{d+1} and v_{d+4} since $|(d+1) - (-(d+3))| = 2d+4 \equiv -4 \pmod{2d+8} \in \langle S \rangle$ and $|(d+4) - (-(d+3))| = 2d+7 \equiv -1 \pmod{2d+8} \in \langle S \rangle$. Next, we note that $v_{d+4} \sim v_{d+3}$ since $|(d+4) - (d+3)| = 1 \in S$, and $v_{d+4} \not\sim v_{d+2}$ since $|(d+4) - (d+2)| = 2 \notin S$. Also note that $v_{-(d+3)}$ is adjacent to neither v_{d+2} nor v_{d+3} since $|(d+2) - (-(d+3))| = 2d+5 \equiv -3 \pmod{2d+8} \notin \langle S \rangle$ and $|(d+3) - (-(d+3))| = 2d+6 \equiv -2 \pmod{2d+8} \notin \langle S \rangle$. Furthermore, v_{d+1} is adjacent to neither v_{d+3} nor v_{d+4} since $|(d+3) - (d+1)| = 2 \notin S$ and $|(d+4) - (d+1)| = 3 \notin S$. Hence, H_w is isomorphic to C_5 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.4.4.2 $w = v_{-(d+1)}$.

Let $H'_w = H[\{v_2, v_3\}]$ and $H''_w = H[\{v_{-(d+3)}, v_{d+4}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+3)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$ and $|3 - (-(d+3))| = d+6 \notin \langle S \rangle$. Similarly, v_{d+4} is adjacent to neither v_2 nor v_3 since $|(d+4) - 2| = d+2 \notin \langle S \rangle$ and $|(d+4) - 3| = d+1 \notin \langle S \rangle$.

We now note that $v_{-(d+3)} \sim v_{d+4}$ since $|(d+4) - (-(d+3))| = 2d+7 \equiv -1 \pmod{2d+8} \in \langle S \rangle$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.4.4.3 $w = v_{-(d+2)}$.

We first note that $V(H_w) = \{v_2, v_3, v_{d+3}, v_{d+4}\}$. We claim that H_w is isomorphic to P_4 . Observe that v_{d+3} is adjacent to both v_3 and v_{d+4} since $|(d+3) - 3| = d \in S$

and $|(d+4) - (d+3)| = 1 \in S$. Also note that v_2 is adjacent to neither v_{d+3} nor v_{d+4} since $|(d+3) - 2| = d+1 \notin \langle S \rangle$ and $|(d+4) - 2| = d+2 \notin \langle S \rangle$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and $v_3 \not\sim v_{d+4}$ since $|(d+4) - 3| = d+1 \notin \langle S \rangle$. Hence, H_w is isomorphic to P_4 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.4.4.4 $w = v_3$ or $w = v_{-3}$.

By symmetry, we need only examine $w = v_{-3}$. Observe $V(H_w) = \{v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. We claim that H_w is isomorphic to P_4 . Observe that v_{d+3} is adjacent to both v_{d+2} and v_{d+4} since $|(d+3) - (d+2)| = 1 \in S$ and $|(d+4) - (d+3)| = 1 \in S$. Next, we note that v_{d+1} is adjacent to neither v_{d+3} nor v_{d+4} since $|(d+3) - (d+1)| = 2 \notin S$ and $|(d+4) - (d+1)| = 3 \notin S$. Furthermore, $v_{d+2} \sim v_{d+1}$ since $|(d+2) - (d+1)| = 1 \in S$, and $v_{d+2} \not\sim v_{d+4}$ since $|(d+4) - (d+2)| = 2 \notin S$. Hence, H_w is isomorphic to P_4 , and thus is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.4.4.

Case 5.4.5 $d \geq 7$ and $n = 2d + 9$.

Note that $V(H) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{-2}$. Since v_{-2} is adjacent to v_{-2+j} and v_{-2-j} for each j in the set S , it follows that $N_H[v_{-2}] = \{v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3\}$. We will show for each $w \in N_H[v_{-2}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.4.5.1 $w = v_{-2}$ or $w = v_2$.

By symmetry, we need only examine $w = v_{-2}$. In this case $V(H_w) = \{v_{-(d+4)}, v_{-(d+3)}, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{d+3}$. Since v_{d+3} is adjacent to $v_{(d+3)+j}$ and

$v_{(d+3)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{d+3}] = \{v_{d+2}, v_{d+3}, v_{d+4}\}$. We will show for each $u \in N_{H_w}[v_{d+3}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.4.5.1.1 $u = v_{d+3}$.

In this case $V(H_u) = \{v_{-(d+4)}, v_{-(d+3)}, v_{d+1}\}$. Observe that $v_{-(d+4)} \sim v_{-(d+3)}$ since $|-(d+3) - (-(d+4))| = 1 \in S$. Next, we note that v_{d+1} is adjacent to both $v_{-(d+4)}$ and $v_{-(d+3)}$ since $|(d+1) - (-(d+4))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$ and $|(d+1) - (-(d+3))| = 2d+4 \equiv -5 \pmod{2d+9} \in \langle S \rangle$. Hence, H_u is isomorphic to C_3 , and thus is well-covered with $\beta(H_u) = 1$.

Case 5.4.5.1.2 $u = v_{d+4}$.

In this case $V(u) = \{v_{-(d+3)}, v_{d+1}, v_{d+2}\}$. Observe that $v_{-(d+3)}$ is adjacent to both v_{d+1} and v_{d+2} since $|(d+1) - (-(d+3))| = 2d+4 \equiv -5 \pmod{2d+9} \in \langle S \rangle$ and $|(d+2) - (-(d+3))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$. Furthermore, $v_{d+1} \sim v_{d+2}$ since $|(d+2) - (d+1)| = 1 \in S$. Hence, H_u is isomorphic to C_3 , and thus is well-covered with $\beta(H_u) = 1$.

Case 5.4.5.1.3 $u = v_{d+2}$.

In this case $V(H_u) = \{v_{-(d+4)}, v_{d+4}\}$. Observe that $v_{d+4} \sim v_{-(d+4)}$ since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \in \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.4.5.1.

Case 5.4.5.2 $w = v_{-(d+1)}$.

Let $H'_w = H[\{v_2, v_3\}]$ and $H''_w = H[\{v_{-(d+4)}, v_{-(d+3)}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+3)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$ and $|3 - (-(d+3))| = d+6 \notin \langle S \rangle$. Similarly, $v_{-(d+4)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+4))| = d+6 \notin \langle S \rangle$ and $|3 - (-(d+4))| = d+7 \notin \langle S \rangle$.

We now note that $v_{-(d+3)} \sim v_{-(d+4)}$ since $|-(d+3) - (-(d+4))| = 1 \in S$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.4.5.3 $w = v_{-(d+2)}$.

Let $H'_w = H[\{v_2, v_3\}]$ and $H''_w = H[\{v_{-(d+4)}, v_{d+4}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+4)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+4))| = d+6 \notin \langle S \rangle$ and $|3 - (-(d+4))| = d+7 \notin \langle S \rangle$. Similarly, v_{d+4} is adjacent to neither v_2 nor v_3 since $|(d+4) - 2| = d+2 \notin \langle S \rangle$ and $|(d+4) - 3| = d+1 \notin \langle S \rangle$.

We now note that $v_{-(d+4)} \sim v_{d+4}$ since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \in \langle S \rangle$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.4.5.4 $w = v_3$ or $w = v_{-3}$.

By symmetry, we need only examine $w = v_{-3}$. Note that $V(H_w) = \{v_{-(d+4)}, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$.

We claim that H_w is isomorphic to C_5 . Observe that $v_{-(d+4)}$ is adjacent to neither v_{d+2} nor v_{d+3} since $|(d+2) - (-(d+4))| = 2d+6 \equiv -3 \pmod{2d+9} \notin \langle S \rangle$ and

$|(d+3) - (-(d+4))| = 2d+7 \equiv -2 \pmod{2d+9} \notin \langle S \rangle$. Next, we note that $v_{-(d+4)}$ is adjacent to both v_{d+1} and v_{d+4} since $|(d+1) - (-(d+4))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$ and $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \in \langle S \rangle$. Furthermore, from Case 5.4.4.1, we know that v_{d+2} is adjacent to both v_{d+1} and v_{d+3} ; v_{d+1} is adjacent to neither v_{d+3} nor v_{d+4} ; $v_{d+4} \sim v_{d+3}$; and $v_{d+4} \not\sim v_{d+2}$. Hence, H_w is isomorphic to C_5 , and thus is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.4.5.

Case 5.4.6 $d \geq 6$ and $2d+8 \leq n \leq 3d+2$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{-2}$. Since v_{-2} is adjacent to v_{-2+j} and v_{-2-j} for each j in the set S , it follows that $N_H[v_{-2}] = \{v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3\}$. We will show for each $w \in N_H[v_{-2}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.4.6.1 $w = v_2$ or $w = v_{-2}$.

By symmetry, we need only examine $w = v_{-2}$. Note that $V(H_w) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+3)\} \cup \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{d+1}$. Since v_{d+1} is adjacent to v_{d+1+j} and v_{d+1-j} for each j in the set S , it follows that $N_{H_w}[v_{d+1}] = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+3)\} \cup \{v_{d+1}, v_{d+2}\} \cup \{v_i : d+5 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We will show for each $u \in N_{H_w}[v_{d+1}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 1$.

Case 5.4.6.1.1 $u = v_{d+1}$.

In this case $V(H_u) = \{v_{d+3}, v_{d+4}\}$. Observe that $v_{d+3} \sim v_{d+4}$ since $|(d+4) - (d+3)| = 1 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.4.6.1.2 $u = v_{d+2}$.

In this case $V(u) = \{v_{d+4}, v_{d+5}\}$. Observe that $v_{d+5} \sim v_{d+4}$ since $|(d+5) - (d+4)| = 1 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.4.6.1.3 $u = v_k$ or $u = v_{-k}$ for $d+5 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By symmetry, we need only examine $u = v_k$. Since neither 2 nor 3 is in S , we can deduce that v_k is adjacent to neither v_{k-3} , v_{k-2} , v_{k+2} nor v_{k+3} , and thus $V(H_u) = \{v_{k-3}, v_{k-2}, v_{k+2}, v_{k+3}\}$. We claim that H_u is isomorphic to K_4 . Observe that v_{k+2} is adjacent to v_{k-3} , v_{k-2} and v_{k+3} since $|(k+2) - (k-3)| = 5 \in S$, $|(k+2) - (k-2)| = 4 \in S$ and $|(k+3) - (k+2)| = 1 \in S$. Next, we note that v_{k+3} is adjacent to both v_{k-3} and v_{k-2} since $|(k+3) - (k-3)| = 6 \in S$ and $|(k+3) - (k-2)| = 5 \in S$. Furthermore, $v_{k-2} \sim v_{k-3}$ since $|(k-2) - (k-3)| = 1 \in S$. Hence, H_u is isomorphic to K_4 , and thus is well-covered with $\beta(H_u) = 1$.

Case 5.4.6.1.4 $u = v_{-(d+4)}$.

In this case $V(H_u) = \{v_{-(d+7)}, v_{-(d+6)}\}$. Observe that $v_{-(d+7)} \sim v_{-(d+6)}$ since $|-(d+6) - (-(d+7))| = 1 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

Case 5.4.6.1.5 $u = v_{-(d+3)}$.

In this case $V(H_u) = \{v_{-(d+6)}, v_{-(d+5)}\}$. Observe that $v_{-(d+6)} \sim v_{-(d+5)}$ since $|-(d+5) - (-(d+6))| = 1 \in S$. Hence, H_u is well-covered with $\beta(H_u) = 1$.

This concludes Case 5.4.6.1.

Case 5.4.6.2 $w = v_{-(d+1)}$.

Let $H'_w = H[\{v_2, v_3\}]$ and $H''_w = H[\{v_{-(d+4)}, v_{-(d+3)}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+3)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$ and $|3 - (-(d+3))| = d+6 \notin \langle S \rangle$. Similarly, $v_{-(d+4)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+4))| = d+6 \notin \langle S \rangle$ and $|3 - (-(d+4))| = d+7 \notin \langle S \rangle$.

We now note that $v_{-(d+3)} \sim v_{-(d+4)}$ since $|-(d+3) - (-(d+4))| = 1 \in S$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.4.6.3 $w = v_{-(d+2)}$.

Let $H'_w = H[\{v_2, v_3\}]$ and $H''_w = H[\{v_{-(d+5)}, v_{-(d+4)}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+5)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+5))| = d+7 \notin \langle S \rangle$ and $|3 - (-(d+5))| = d+8 \notin \langle S \rangle$. Similarly, $v_{-(d+4)}$ is adjacent to neither v_2 nor v_3 since $|2 - (-(d+4))| = d+6 \notin \langle S \rangle$ and $|3 - (-(d+4))| = d+7 \notin \langle S \rangle$.

We now note that $v_{-(d+4)} \sim v_{-(d+5)}$ since $|-(d+4) - (-(d+5))| = 1 \in S$, and thence H''_w is well-covered with $\beta(H''_w) = 1$. Furthermore, $v_2 \sim v_3$ since $|3 - 2| = 1 \in S$, and thence H'_w is well-covered with $\beta(H'_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 2$.

Case 5.4.6.4 $w = v_3$ or $w = v_{-3}$.

By symmetry, we need only examine $w = v_3$. Note that $V(H_w) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_i : d+4 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $x_0 = v_{-(d+1)}$. Since $v_{-(d+1)}$ is adjacent to $v_{-(d+1)+j}$ and $v_{-(d+1)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+1)}] =$

$\{v_i: -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+5)\} \cup \{v_{-(d+2)}, v_{-(d+1)}\} \cup \{v_i: d+4 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. We will show for each $x \in N_{H_w}[v_{-(d+1)}]$ that $H_x = H_w \setminus N_{H_w}[x]$ is well-covered with $\beta(H_x) = 1$.

Case 5.4.6.4.1 $x = v_{-(d+1)}$.

In this case $V(H_x) = \{v_{-(d+4)}, v_{-(d+3)}\}$. Observe that $v_{-(d+3)} \sim v_{-(d+4)}$ since $|-(d+3) - (-(d+4))| = 1 \in S$. Hence, H_x is well-covered with $\beta(H_x) = 1$.

Case 5.4.6.4.2 $x = v_{-(d+2)}$.

In this case $V(H_x) = \{v_{-(d+5)}, v_{-(d+4)}\}$. Observe that $v_{-(d+4)} \sim v_{-(d+5)}$ since $|-(d+4) - (-(d+5))| = 1 \in S$. Hence, H_x is well-covered with $\beta(H_x) = 1$.

Case 5.4.6.4.3 $x = v_k$ or $x = v_{-k}$ for $d+5 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By symmetry, we need only examine $x = v_k$. Since neither 2 nor 3 is in S , we can deduce that v_k is adjacent to neither v_{k-3} , v_{k-2} , v_{k+2} nor v_{k+3} , and thus $V(H_x) = \{v_{k-3}, v_{k-2}, v_{k+2}, v_{k+3}\}$. From Case 5.4.6.1.3, we know that H_x is isomorphic to K_4 , and thus is well-covered with $\beta(H_x) = 1$.

Case 5.4.6.4.4 $x = v_{d+4}$.

In this case $V(H_x) = \{v_{d+6}, v_{d+7}\}$. Observe that $v_{d+6} \sim v_{d+7}$ since $|(d+7) - (d+6)| = 1 \in S$. Hence, H_x is well-covered with $\beta(H_x) = 1$.

This concludes Case 5.4.6.4.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case 5.4.6.

(iv) G is one of the following: $C(10, \{1, 4, 5\})$, $C(11, \{1, 4, 5\})$, $C(8, \{1, 4\})$, $C(10, \{1, 4\})$, $C(11, \{1, 4\})$, $C(12, \{1, 4\})$, or $C(13, \{1, 4\})$.

Note that $C(10, \{1, 4, 5\})$ and $C(11, \{1, 4, 5\})$ are well-covered and $\beta(G) = 2$; $C(8, \{1, 4\})$ is well-covered and $\beta(G) = 3$; $C(10, \{1, 4\})$, $C(11, \{1, 4\})$ and $C(12, \{1, 4\})$ are well-covered and $\beta(G) = 4$; and $C(13, \{1, 4\})$ is well-covered and $\beta(G) = 5$.

We now proceed to prove the ‘only if’ direction. First, observe that $C(9, \{1, 4\})$ is not well-covered. Next, we consider the remaining cases.

Case 5.4.7 $d = 4$ and $n \geq 14$.

Observe that for each $14 \leq n \leq 18$ one can verify that G is not well-covered.

Consider the case where $n \geq 19$. Let $I' = \{v_{-5}, v_{-2}, v_7, v_9\}$. Observe that v_{-2} is adjacent to neither v_{-5} nor v_7 since $|(-2) - (-5)| = 3 \notin S$ and $|7 - (-2)| = 9 \notin S$. Also note that $v_7 \not\sim v_9$ since $|9 - 7| = 2 \notin S$. Next, we consider v_{-5} . Note that $|7 - (-5)| = 12$ and $|9 - (-5)| = 14$. Given our assumption that $n \geq 19$, it follows that $n - 12 \geq 7$ and $n - 14 \geq 5$. Hence, v_{-5} is adjacent to neither v_7 nor v_9 . Finally, we show that $v_{-2} \not\sim v_9$. Note that $|9 - (-2)| = 11$. Given our assumption that $n \geq 19$, it follows that $n - 11 \geq 8$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_0, v_1, v_4\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_1, v_4\}$. Note that $v_1 \not\sim v_4$ since $|4 - 1| = 3 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.4.8 $d = 5$ and $n \geq 12$.

Observe that for each $12 \leq n \leq 29$ one can verify that G is not well-covered.

Consider the case where $n \geq 30$. Let $I' = \{v_{-12}, v_{-5}, v_5, v_{12}\}$. Observe that v_5 is adjacent to neither v_{-5} nor v_{12} since $|5 - (-5)| = 10 \notin S$ and $|12 - 5| = 7 \notin S$. Next, we consider v_{12} . Note that $|12 - (-5)| = 17$ and $|12 - (-12)| = 24$. Given our assumption that $n \geq 30$, it follows that $n - 17 \geq 13$ and $n - 24 \geq 6$. Hence, v_{12} is adjacent to neither v_{-5} nor v_{-12} . By symmetry, we can also deduce that v_{-12} is adjacent to neither v_{-5} nor v_5 . Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_2 . It follows that $V(H_1) = \{v_{-3}, v_{-2}, v_2, v_3\}$. First, let $K_1 = \{v_2\}$. Note that v_2 is adjacent to v_{-3} , v_2 and v_3 since $|2 - (-3)| = 5 \in S$, $|2 - (-2)| = 4 \in S$ and $|3 - 2| = 1 \in S$. Hence, K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_3\}$. Observe that $v_3 \not\sim v_{-3}$ since $|3 - (-3)| = 6 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.4.9 $d \geq 6$ and $n = 2d + 4$.

Then $V(H) = \{v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}\}$. Let $I_1 = \{v_0, v_{d+2}\}$. Observe that $v_0 \not\sim v_{d+2}$ since $|(d+2) - 0| = d+2 \notin \langle S \rangle$. Consider any vertex $v_i \in V(H)$ with $v_i \neq v_{d+2}$. We claim that $v_i \sim v_{d+2}$. Note that v_{d+2} is adjacent to v_2 , v_3 and v_{d+1} since $|(d+2) - 2| = d \in S$, $|(d+2) - 3| = d-1 \in S$ and $|(d+2) - (d+1)| = 1 \in S$. By symmetry, we can also deduce that v_{d+2} is adjacent to $v_{-(d+1)}$, v_{-3} and v_{-2} . Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_{d+1}, v_{-(d+1)}\}$. Observe that v_0 is adjacent to neither $v_{-(d+1)}$ nor v_{d+1} since $|0 - (-(d+1))| = d+1 \notin \langle S \rangle$ and $|(d+1) - 0| = d+1 \notin \langle S \rangle$. Furthermore, $v_{d+1} \not\sim v_{-(d+1)}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -2 \pmod{2d+4} \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 5.4.10 $d \geq 6$ and $n = 2d + 6$.

Then $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}, v_{d+3}\}$. Let $I_1 = \{v_0, v_{d+2}, v_{-(d+2)}\}$. Observe that v_0 is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} since $|0 - (-(d+2))| = d + 2 \notin \langle S \rangle$ and $|(d+2) - 0| = d + 2 \notin \langle S \rangle$. Next, we note that $v_{d+2} \sim v_{d+3}$ since $|(d+3) - (d+2)| = 1 \in S$, and $v_{-(d+2)} \sim v_{-(d+1)}$ since $|-(d+1) - (-(d+2))| = 1 \in S$. Also note that $v_{d+2} \not\sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -2 \pmod{2d+6} \notin \langle S \rangle$, and $v_{-(d+2)}$ is adjacent to v_{-2} and v_{-3} since $|-2 - (-(d+2))| = d \in S$ and $|-3 - (-(d+2))| = d - 1 \in S$. Furthermore, from Case 5.4.9, we know that v_{d+2} is adjacent to v_2, v_3 and v_{d+1} . Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_2, v_{d+3}, v_{-(d+1)}\}$. Observe that v_0 is adjacent to neither $v_{-(d+1)}$, v_2 nor v_{d+3} since $|0 - (-(d+1))| = d + 1 \notin \langle S \rangle$, $|2 - 0| = 2 \notin S$ and $|(d+3) - 0| = d + 3 \notin \langle S \rangle$. Next, we note that v_2 is adjacent to neither $v_{-(d+1)}$ nor v_{d+3} since $|2 - (-(d+1))| = d + 3 \notin \langle S \rangle$ and $|(d+3) - 2| = d + 1 \notin \langle S \rangle$. Furthermore, $v_{d+3} \not\sim v_{-(d+1)}$ since $|(d+3) - (-(d+1))| = 2d + 4 \equiv -2 \pmod{2d+6} \notin \langle S \rangle$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 5.4.11 $d \geq 6$ and $n = 2d + 7$.

Then $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-3}, v_{-2}, v_2, v_3, v_{d+1}, v_{d+2}, v_{d+3}\}$. Let $I_1 = \{v_0, v_{d+2}, v_{-(d+2)}\}$. Observe that $v_{d+2} \not\sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -3 \pmod{2d+7} \notin \langle S \rangle$. Next, we note that $v_{-(d+2)} \sim v_{-(d+3)}$ since $|-(d+2) - (-(d+3))| = 1$. Furthermore, from Case 5.4.9 and 5.4.10, we know that v_0 is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} ; v_{d+2} is adjacent to v_2, v_3, v_{d+1} and v_{d+3} ; and $v_{-(d+2)}$ is adjacent to v_{-2}, v_{-3} and $v_{-(d+1)}$. Hence, I_1 is a maximal independent set in G .

Next, let $I_2 = \{v_0, v_2, v_{d+3}, v_{-(d+2)}\}$. Observe that $v_{-(d+2)}$ is adjacent to neither v_0 nor v_2 since $|0 - (-(d+2))| = d + 2 \notin \langle S \rangle$ and $|2 - (-(d+2))| = d + 4 \notin \langle S \rangle$. Next, we note that $v_{d+3} \not\sim v_{-(d+2)}$ since $|(d+3) - (-(d+2))| = 2d+5 \equiv -2 \pmod{2d+7} \notin \langle S \rangle$.

Furthermore, from Case 5.4.10, we know that v_0 is adjacent to neither v_2 nor v_{d+3} and $v_2 \not\sim v_{d+3}$. Hence, I_2 is an independent set in G with cardinality greater than that of I_1 , and thus G is not well-covered.

Case 5.4.12 $d \geq 6$ and $n \geq 3d + 3$.

Case 5.4.12.1 $n = 3d + 3$.

Let $I' = \{v_{-(d+1)}, v_{d+1}\}$. Note that $v_{-(d+1)} \not\sim v_{d+1}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+1) \pmod{3d+3} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_{-(d-2)}, v_0, v_{d-2}, v_{d-1}\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-(d-2)}, v_{d-2}\}$. Note that $v_{-(d-2)} \not\sim v_{d-2}$ since $|(d-2) - (-(d-2))| = 2d-4 \equiv -(d+7) \pmod{3d+3} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 4$ is also not well-covered.

Case 5.4.12.2 $n = 3d + 5$.

Let $I' = \{v_{-(d+2)}, v_{d+2}\}$. Note that $v_{-(d+2)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+1) \pmod{3d+5} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_{-d}, v_{-1}, v_0, v_1, v_{d-1}, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-(d-1)}, v_{d-1}\}$. Note that $v_{-(d-1)} \not\sim v_{d-1}$ since $|(d-1) - (-(d-1))| = 2d-2 \equiv -(d+7) \pmod{3d+5} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 6$ is also not well-covered.

Case 5.4.12.3 $n = 3d + 7$.

Let $I' = \{v_{-(d+2)}, v_{d+4}, v_{d+2}\}$. Observe that $v_{d+4} \not\sim v_{d+2}$ since $|(d+4) - (d+2)| = 2 \notin S$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{d+4} nor v_{d+2} since $|(d+4) - (-(d+2))| = 2d+6 \equiv -(d+1) \pmod{3d+7} \notin \langle S \rangle$ and $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+3) \pmod{3d+7} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_{-d}, v_{-1}, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Note that $v_1 \not\sim v_{-1}$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 8$ is also not well-covered.

Case 5.4.12.4 $n \geq 3d + 9$.

Let $I' = \{v_{-(d+4)}, v_{-(d+2)}, v_{d+2}, v_{d+4}\}$. Observe that $v_{d+4} \not\sim v_{d+2}$ since $|(d+4) - (d+2)| = 2 \notin S$. Next, we show that $v_{d+4} \not\sim v_{-(d+4)}$. Note that $|(d+4) - (-(d+4))| = 2d+8$. Given our assumption that $n \geq 3d+9$, it follows that $n - (2d+8) \geq d+1$. Finally, we consider $v_{-(d+2)}$. Note that $|(d+4) - (-(d+2))| = 2d+6$ and $|(d+2) - (-(d+2))| = 2d+4$. Given our assumption that $n \geq 3d+9$, it follows that $n - (2d+6) \geq d+3$ and $n - (2d+4) \geq d+5$. Hence, $v_{-(d+2)}$ is adjacent to neither v_{d+4} nor v_{d+2} . By symmetry, we can also deduce that $v_{-(d+4)}$ is adjacent to neither $v_{-(d+2)}$ nor v_{d+2} .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-1}, v_0, v_1\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-1}, v_1\}$. Note that $v_1 \not\sim v_{-1}$ since $|1 - (-1)| = 2 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. ■

A characterization of the well-covered graphs in Class 10 can now be stated.

Theorem 5.5 *Let n and d be integers with $3 \leq d \leq \frac{n}{2}$. Then $G = C(n, \{3, 4, \dots, d\})$ is well-covered if and only if one of the following conditions holds:*

- (i) $d \geq 3$ and either $n = 2d$ or $n = 2d + 1$, or
- (ii) $d \geq 3$ and $n = 2d + 3$, or
- (iii) $d \geq 4$ and $n = 2d + 4$, or
- (iv) $d \geq 4$ and $n = 2d + 5$, or
- (v) $d \geq 3$ and $n = 2d + 6$, or
- (vi) $d \geq 4$ and $2d + 7 \leq n \leq 3d + 2$, or
- (vii) $d \geq 3$ and $n = 3d + 6$, or
- (viii) $G = C(21, \{3\})$.

Furthermore, if (i) or (ii) holds $\beta(G) = 3$; if (iii) holds $\beta(G) = 4$; if (iv) holds $\beta(G) = 5$; if (v), (vi) or (vii) holds $\beta(G) = 6$; and if (viii) holds $\beta(G) = 9$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. First, we prove the ‘if’ direction. In each case let I be a maximal independent set of G . Without loss of generality, assume that $v_0 \in I$ and let H be the graph induced by $G \setminus N[v_0]$. Then $V(H) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d + 1)\} \cup \{v_{-2}, v_{-1}\} \cup \{v_1, v_2\} \cup \{v_i : d + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. By Corollary 2.6, it suffices to show that in Case (ii) H is well-covered with $\beta(H) = 2$; in Case (iii) H is well-covered with $\beta(H) = 3$; in Case (iv) H is well-covered with $\beta(H) = 4$; and in Case (v), (vi) or (vii) H is well-covered with $\beta(H) = 5$.

(i) $d \geq 3$ and either $n = 2d$ or $n = 2d + 1$.

In this case, the theorem follows as a consequence of Theorem 3.4. Furthermore,
 $\beta(G) = 3$.

(ii) $d \geq 3$ and $n = 2d + 3$.

First, when $d = 3$ we note that $C(9, \{3\})$ is well-covered and $\beta(G) = 3$.

Next, we consider the case where $d \geq 4$. Note that $V(H) = \{v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 1$.

Case 5.5.1 $w = v_1$.

In this case $V(H_w) = \{v_{-(d+1)}, v_{-1}, v_2\}$. Note that v_{-1} is adjacent to both $v_{-(d+1)}$ and v_2 since $|(-1) - (-(d+1))| = d \in S$ and $|2 - (-1)| = 3 \in S$. In addition, $v_{-(d+1)} \sim v_2$ since $|2 - (-(d+1))| = d + 3 \equiv -d \pmod{2d+3} \in \langle S \rangle$. Hence, H_w is isomorphic to C_3 , and thus is well-covered with $\beta(H_w) = 1$.

Case 5.5.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-(d+1)}, v_{-1}\}$. Note that $v_{-(d+1)} \sim v_{-1}$. Hence, H_w is well-covered with $\beta(H_w) = 1$.

Case 5.5.3 $w = v_{-2}$.

In this case $V(H_w) = \{v_{-1}\}$. Hence, H_w is well-covered with $\beta(H_w) = 1$.

Hence, G is well-covered with $\beta(G) = 3$, concluding the proof of Case (ii).

(iii) $d \geq 4$ and $n = 2d + 4$.

In this case $V(H) = \{v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 2$.

Case 5.5.4 $w = v_1$.

In this case $V(H_w) = \{v_{-(d+1)}, v_{-1}, v_2, v_{d+2}\}$. We claim that H_w is isomorphic to P_4 . Observe that $v_{-(d+1)}$ is adjacent to neither v_2 nor v_{d+2} since $|2 - (-(d+1))| = d + 3 \equiv -(d+1) \pmod{2d+4} \notin \langle S \rangle$ and $|(d+2) - (-(d+1))| = 2d + 3 \equiv -1 \pmod{2d+4} \notin \langle S \rangle$. Next, we note that $v_2 \sim v_{d+2}$ since $|(d+2) - 2| = d \in S$, and $v_{-1} \not\sim v_{d+2}$ since $|(d+2) - (-1)| = d + 3 \equiv -(d+1) \pmod{2d+4} \notin \langle S \rangle$. Furthermore, from Case (ii), we know that v_{-1} is adjacent to both $v_{-(d+1)}$ and v_2 . Hence, H_w is isomorphic to P_4 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.5.5 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-(d+1)}, v_{-2}, v_{-1}, v_{d+2}\}$. We claim that H_w is isomorphic to P_4 . Observe that $v_{-2} \sim v_{-(d+1)}$ since $|(-2) - (-(d+1))| = d - 1 \in S$, and $v_{-(d+1)} \not\sim v_{d+2}$ since $|(d+2) - (-(d+1))| = 2d + 3 \equiv -1 \pmod{2d+4} \notin \langle S \rangle$. Next, we note that $v_{-2} \sim v_{d+2}$ since $|(d+2) - (-2)| = d + 4 \equiv -d \pmod{2d+4} \in \langle S \rangle$. We also note that v_{-1} is adjacent to neither v_{-2} nor v_{d+2} since $|(-1) - (-2)| = 1 \notin S$ and $|(d+2) - (-1)| = d + 3 \equiv -(d+1) \pmod{2d+4} \notin \langle S \rangle$. Furthermore, from Case (ii), we know that $v_{-1} \sim v_{-(d+1)}$. Hence, H_w is isomorphic to P_4 , and thus is well-covered with $\beta(H_w) = 2$.

Case 5.5.6 $w = v_{-2}$.

In this case $V(H_w) = \{v_{-1}, v_{d+1}\}$. Observe that $v_{d+1} \not\sim v_{-1}$ since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$. Hence, H_w is well-covered with $\beta(H_w) = 2$.

Hence, G is well-covered with $\beta(G) = 4$, concluding the proof of Case (iii).

(iv) $d \geq 4$ and $n = 2d + 5$.

In this case $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 3$.

Case 5.5.7 $w = v_1$.

Let $H'_w = H[\{v_{-(d+1)}, v_{-1}, v_2, v_{d+2}\}]$ and $H''_w = H[\{v_{-(d+2)}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Observe that $v_{-(d+2)}$ is adjacent to neither $v_{-(d+1)}$ nor v_{-1} since $|-(d+1) - (-(d+2))| = 1 \notin S$ and $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$. Furthermore, $v_{-(d+2)}$ is adjacent to neither v_2 nor v_{d+2} since $|2 - (-(d+2))| = d+4 \equiv -(d+1) \pmod{2d+5} \notin \langle S \rangle$ and $|(d+2) - (-(d+2))| = 2d+4 \equiv -1 \pmod{2d+5} \notin \langle S \rangle$.

Next, we show that H'_w is isomorphic to P_4 . Observe that $v_{-(d+1)}$ is adjacent to neither v_2 nor v_{d+2} since $|(2) - (-(d+1))| = d+3 \equiv -(d+2) \pmod{2d+5} \notin \langle S \rangle$ and $|(d+2) - (-(d+1))| = 2d+3 \equiv -2 \pmod{2d+5} \notin \langle S \rangle$. Next, we note that $v_{-1} \not\sim v_{d+2}$ since $|(d+2) - (-1)| = d+3 \equiv -(d+2) \pmod{2d+5} \notin \langle S \rangle$. Furthermore, from Case (ii) and (iii), we know that v_{-1} is adjacent to both

$v_{-(d+1)}$ and v_2 ; and $v_2 \sim v_{d+2}$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Finally, we note that H''_w is well-covered with $\beta(H''_w) = 1$. Hence, H_w is well-covered with $\beta(H_w) = 3$.

Case 5.5.8 $w = v_{d+1}$.

Let $H'_w = H[\{v_{-(d+2)}, v_{-2}\}]$, $H''_w = H[\{v_{d+2}\}]$ and $H'''_w = H[\{v_{-1}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ and $V(H'''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w and H'''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{-1} is adjacent to neither $v_{-(d+2)}$ nor v_{-2} since $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$ and $|(-1) - (-2)| = 1 \notin S$. Next, we note that v_{d+2} is adjacent to neither $v_{-(d+2)}$ nor v_{-2} since $|(d+2) - (-(d+2))| = 2d+4 \equiv -1 \pmod{2d+5} \notin \langle S \rangle$ and $|(d+2) - (-2)| = d+4 \equiv -(d+1) \pmod{2d+5} \notin \langle S \rangle$. Furthermore, $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \equiv -(d+2) \pmod{2d+5} \notin \langle S \rangle$.

We now note that $v_{-2} \sim v_{-(d+2)}$ since $|(-2) - (-(d+2))| = d \in S$, hence H'_w is well-covered with $\beta(H'_w) = 1$. Furthermore, H''_w and H'''_w are well-covered with $\beta(H''_w) = \beta(H'''_w) = 1$. Hence, H_w is well-covered with $\beta(H_w) = 3$.

Case 5.5.9 $w = v_{-2}$.

Let $H'_w = H[\{v_{-1}\}]$, $H''_w = H[\{v_{d+1}\}]$ and $H'''_w = H[\{v_{d+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ and $V(H'''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w and H'''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{d+1} is adjacent to neither v_{-1} nor v_{d+2} since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$ and $|(d+2) - (d+1)| = 1 \notin S$. Furthermore, $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \equiv -(d+2) \pmod{2d+5} \notin \langle S \rangle$.

$(\text{mod } 2d + 5) \notin \langle S \rangle$. Hence, H'_w , H''_w and H'''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = \beta(H'''_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 3$.

Hence, G is well-covered with $\beta(G) = 5$, concluding the proof of Case (iv).

(v) $d \geq 3$ and $n = 2d + 6$.

First, when $d = 3$ we note that $C(12, \{3\})$ is well-covered and $\beta(G) = 6$.

Next, we consider the case where $d \geq 4$. Then $V(H) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+3}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.5.10 $w = v_1$.

Let $H'_w = H[\{v_{-(d+1)}, v_{-1}, v_2, v_{d+2}\}]$, $H''_w = H[\{v_{d+3}\}]$ and $H'''_w = H[\{v_{-(d+2)}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ and $V(H'''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w and H'''_w is adjacent to a vertex in either of the other two graphs. Observe that $v_{-(d+2)}$ is adjacent to neither $v_{-(d+1)}$, v_{-1} nor v_2 since $|-(d+1) - (-(d+2))| = 1 \notin S$, $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$ and $|2 - (-(d+2))| = d+4 \notin \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{d+2} nor v_{d+3} since $|(d+2) - (-(d+2))| = 2d+4 \equiv -2 \pmod{2d+6} \notin \langle S \rangle$ and $|(d+3) - (-(d+2))| = 2d+5 \equiv -1 \pmod{2d+6} \notin \langle S \rangle$. Furthermore, v_{d+3} is adjacent to neither $v_{-(d+1)}$, v_{-1} , v_2 nor v_{d+2} since $|(d+3) - (-(d+1))| = 2d+4 \equiv -2 \pmod{2d+6} \notin \langle S \rangle$, $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$, $|(d+3) - 2| = d+1 \notin \langle S \rangle$ and $|(d+3) - (d+2)| = 1 \notin S$.

Next, we will show that H'_w is isomorphic to C_4 . Observe that $v_{-(d+1)} \sim v_{d+2}$ since $|(d+2) - (-(d+1))| = 2d+3 \equiv -3 \pmod{2d+6} \in \langle S \rangle$. Next, we note

that $v_{-1} \not\sim v_{d+2}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$, and $v_{-(d+1)} \not\sim v_2$ since $|(2) - (-(d+1))| = d+3 \notin \langle S \rangle$. Furthermore, from Case (ii) and (iii), we know that v_{-1} is adjacent to both $v_{-(d+1)}$ and v_2 , and $v_2 \sim v_{d+2}$. Hence, H'_w is isomorphic to C_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Finally, H''_w and H'''_w are well-covered with $\beta(H''_w) = \beta(H'''_w) = 1$. Hence, H_1 is well-covered with $\beta(H_1) = 4$.

Case 5.5.11 $w = v_{d+1}$.

Let $H'_w = H[\{v_{-2}\}]$, $H''_w = H[\{v_{-1}\}]$, $H'''_w = H[\{v_{d+2}\}]$ and $H''''_w = H[\{v_{d+3}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$, $V(H'''_w)$ and $V(H''''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w , H'''_w and H''''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{-2} is adjacent to neither v_{-1} , v_{d+2} nor v_{d+3} since $|(-1) - (-2)| = 1 \notin S$, $|(d+2) - (-2)| = d+4 \notin \langle S \rangle$ and $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$. Next, we note that v_{d+3} is adjacent to neither v_{-1} nor v_{d+2} since $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$ and $|(d+3) - (d+2)| = 1 \notin S$. Furthermore, $v_{-1} \not\sim v_{d+2}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, H'_w , H''_w , H'''_w and H''''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = \beta(H'''_w) = \beta(H''''_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.12 $w = v_{-2}$.

Let $H'_w = H[\{v_{-1}\}]$, $H''_w = H[\{v_{d+1}\}]$, $H'''_w = H[\{v_{d+2}\}]$ and $H''''_w = H[\{v_{d+3}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$, $V(H'''_w)$ and $V(H''''_w)$ forms a partition of $V(H_w)$.

Observe that v_{-1} is adjacent to neither v_{d+1} , v_{d+2} nor v_{d+3} since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$. Next,

we note that v_{d+2} is adjacent to neither v_{d+1} nor v_{d+3} since $|(d+2) - (d+1)| = 1 \notin S$ and $|(d+3) - (d+2)| = 1 \notin S$. Furthermore, $v_{d+1} \not\sim v_{d+3}$ since $|(d+3) - (d+1)| = 2 \notin S$. Hence, H'_w, H''_w, H'''_w and H''''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = \beta(H'''_w) = \beta(H''''_w) = 1$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Hence, G is well-covered and $\beta(G) = 6$, concluding the proof of Case (v).

(vi) $d \geq 4$ and $2d + 7 \leq n \leq 3d + 2$.

We consider the following four cases:

Case 5.5.13 $d \geq 5$ and $n = 2d + 7$.

In this case $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+3}\}$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$.

Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that

$N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$

is well-covered with $\beta(H_w) = 4$.

Case 5.5.13.1 $w = v_1$.

Let $H'_w = H[\{v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_2, v_{d+2}, v_{d+3}\}]$ and $H''_w = H[\{v_{-(d+3)}\}]$.

Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in H'_w is adjacent to a vertex in H''_w . Since neither 1

nor 2 is in S , we can deduce that $v_{-(d+3)}$ is adjacent to neither $v_{-(d+2)}, v_{-(d+1)},$

v_{d+2} nor v_{d+3} . Furthermore, $v_{-(d+3)}$ is adjacent to neither v_{-1} nor v_2 since

$|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$ and $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$.

Next, to show that H'_w is well-covered we are going to apply Lemma 5.1 with

$u_0 = v_{d+3}$. Since v_{d+3} is adjacent to $v_{(d+3)+j}$ and $v_{(d+3)-j}$ for each j in the set S ,

it follows that $N_{H'_w}[v_{d+3}] = \{v_{-(d+1)}, v_{d+3}\}$. We will show for each $u \in N_{H'_w}[v_{d+3}]$ that $H_u = H'_w \setminus N_{H'_w}[u]$ is well-covered with $\beta(H_u) = 2$.

Case 5.5.13.1.1 $u = v_{d+3}$.

In this case $V(H_u) = \{v_{-(d+2)}, v_{-1}, v_2, v_{d+2}\}$. We claim that H_u is isomorphic to P_4 . Observe that $v_{d+2} \sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -3 \pmod{2d+7} \notin \langle S \rangle$, and $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Next, we note that v_2 is adjacent to both v_{-1} and v_{d+2} since $|2 - (-1)| = 3 \in S$ and $|(d+2) - 2| = d \in S$. Furthermore, $v_{-(d+2)}$ is adjacent to neither v_{-1} nor v_2 since $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$ and $|2 - (-(d+2))| = d+4 \notin \langle S \rangle$. Hence, H_u is isomorphic to P_4 , and thus is well-covered with $\beta(H_u) = 2$.

Case 5.5.13.1.2 $u = v_{-(d+1)}$.

In this case $V(H_u) = \{v_{-(d+2)}, v_2\}$. Note that $v_{-(d+2)} \not\sim v_2$ since $|2 - (-(d+2))| = d+3 \notin \langle S \rangle$. Hence, H_u is well-covered with $\beta(H_u) = 2$.

Hence, H'_w is well-covered and $\beta(H'_w) = 3$. Furthermore, H''_w is well-covered with $\beta(H''_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.13.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{-1}, v_{d+2}, v_{d+3}\}$. From Case 5.5.11, we know that H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.13.3 $w = v_{-2}$.

Let $H'_w = H[\{v_{-(d+3)}, v_{d+1}\}]$, $H''_w = H[\{v_{d+2}\}]$, $H'''_w = H[\{v_{d+3}\}]$ and $H''''_w = H[\{v_{-1}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$, $V(H'''_w)$ and $V(H''''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w, H''_w, H'''_w and H''''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{-1} is adjacent to neither $v_{-(d+3)}, v_{d+1}, v_{d+2}$ nor v_{d+3} since $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$, $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$. Next, we note that v_{d+3} is adjacent to neither v_{d+1} nor v_{d+2} since $|(d+3) - (d+1)| = 2 \notin S$ and $|(d+3) - (d+2)| = 1 \notin S$; and $v_{d+2} \not\sim v_{d+1}$ since $|(d+2) - (d+1)| = 1 \notin S$. Furthermore, $v_{-(d+3)}$ is adjacent to neither v_{d+2} nor v_{d+3} since $|(d+2) - (-(d+3))| = 2d+5 \equiv -2 \pmod{2d+7} \notin \langle S \rangle$ and $|(d+3) - (-(d+3))| = 2d+6 \equiv -1 \pmod{2d+7} \notin \langle S \rangle$.

Next, we note that $v_{-(d+3)} \sim v_{d+1}$ since $|(d+1) - (-(d+3))| = 2d+4 \equiv -3 \pmod{2d+7} \in \langle S \rangle$. Hence, H'_w is well-covered with $\beta(H'_w) = 1$. Furthermore, H''_w, H'''_w , and H''''_w are well-covered with $\beta(H''_w) = \beta(H'''_w) = \beta(H''''_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Hence, G is well-covered and $\beta(G) = 6$, concluding the proof of Case 5.5.13.

Case 5.5.14 $d \geq 6$ and $n = 2d + 8$.

In this case $V(H) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.5.14.1 $w = v_1$.

In this case $V(H_w) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_2, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{d+3}$. Since v_{d+3} is adjacent to $v_{(d+3)+j}$ and $v_{(d+3)-j}$ for each j in the set S , it follows that

$N_{H_w}[v_{d+3}] = \{v_{-(d+2)}, v_{-(d+1)}, v_{d+3}\}$. We will show for each $u \in N_{H_w}[v_{d+3}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 3$.

Case 5.5.14.1.1 $u = v_{d+3}$.

Let $H'_u = H_w[\{v_{-(d+3)}, v_{-1}, v_2, v_{d+2}\}]$ and $H''_u = H_w[\{v_{d+4}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in H'_u is adjacent to a vertex in H''_u . Note that $v_{-(d+3)} \not\sim v_{d+4}$ since $|(d+4) - (-(d+3))| = 2d+7 \equiv -1 \pmod{2d+8} \notin \langle S \rangle$. Furthermore, v_{d+4} is adjacent to neither v_{-1} , v_2 nor v_{d+2} since $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$, $|(d+4) - (2)| = d+2 \notin \langle S \rangle$ and $|(d+4) - (d+2)| = 2 \notin S$.

Next, we show that H'_u is isomorphic to P_4 . Observe that $v_{d+2} \sim v_{-(d+3)}$ since $|(d+2) - (-(d+3))| = 2d+5 \equiv -3 \pmod{2d+8} \in \langle S \rangle$. Next, we note that v_2 is adjacent to both v_{-1} and v_{d+2} since $|2 - (-1)| = 3 \in S$ and $|(d+2) - 2| = d \in S$. Also note that v_{-1} is adjacent to neither $v_{-(d+3)}$ nor v_{d+2} since $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$ and $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Furthermore, $v_2 \not\sim v_{-(d+3)}$ since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, H''_u is well-covered with $\beta(H''_u) = 1$. Hence, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.5.14.1.2 $u = v_{-(d+1)}$.

In this case $V(H_u) = \{v_{-(d+3)}, v_{-(d+2)}, v_2\}$. Note that v_2 is adjacent to neither $v_{-(d+3)}$ nor $v_{-(d+2)}$ since $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$ and $|2 - (-(d+2))| = d+4 \notin \langle S \rangle$. Furthermore, $v_{-(d+3)} \not\sim v_{-(d+2)}$ since $|-(d+2) - (-(d+3))| = 1 \notin S$. Hence, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.5.14.1.3 $u = v_{-(d+2)}$.

Let $H'_u = H_w[\{v_{-(d+1)}, v_{-1}, v_2, v_{d+4}\}]$ and $H''_u = H_w[\{v_{-(d+3)}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in H'_u is adjacent to a vertex in H''_u . Note that $v_{-(d+3)}$ is adjacent to neither $v_{-(d+1)}, v_{-1}, v_2$ nor v_{d+4} since $|(-(d+1)) - (-(d+3))| = 2 \notin S$, $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$, $|2 - (-(d+3))| = d+5 \notin \langle S \rangle$ and $|(d+4) - (-(d+3))| = 2d+7 \equiv -1 \pmod{2d+8} \notin \langle S \rangle$.

Next, we show that H'_u is isomorphic to P_4 . Observe that v_{-1} is adjacent to both $v_{-(d+1)}$ and v_2 since $|(-1) - (-(d+1))| = d \in S$ and $|2 - (-1)| = 3 \in S$. Next, we note that v_2 is adjacent to neither $v_{-(d+1)}$ nor v_{d+4} since $|2 - (-(d+1))| = d+3 \notin \langle S \rangle$ and $|(d+4) - 2| = d+2 \notin \langle S \rangle$. Furthermore, $v_{d+4} \sim v_{-(d+1)}$ since $|(d+4) - (-(d+1))| = 2d+5 \equiv -3 \pmod{2d+8} \in \langle S \rangle$, and $v_{-1} \not\sim v_{d+4}$ since $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, H''_u is well-covered with $\beta(H''_u) = 1$. Hence, H_u is well-covered with $\beta(H_u) = 3$.

This concludes Case 5.5.14.1.

Case 5.5.14.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{-1}, v_{d+2}, v_{d+3}\}$. From Case 5.5.11, we know that H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.14.3 $w = v_{-2}$.

Let $H'_w = H[\{v_{-(d+3)}, v_{d+1}, v_{d+2}, v_{d+4}\}]$, $H''_w = H[\{v_{-1}\}]$ and $H'''_w = H[\{v_{d+3}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ and $V(H'''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w and H'''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{-1} is adjacent to neither $v_{-(d+3)}$, v_{d+1} , v_{d+2} , nor v_{d+4} since $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$, $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$ and $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Next, we note that v_{d+3} is adjacent to neither v_{d+1} , v_{d+2} nor v_{d+4} since $|(d+3) - (d+1)| = 2 \notin S$, $|(d+3) - (d+2)| = 1 \notin S$ and $|(d+4) - (d+3)| = 1 \notin S$. Furthermore, $v_{-(d+3)} \not\sim v_{d+3}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -2 \pmod{2d+8} \notin \langle S \rangle$, and $v_{d+3} \not\sim v_{-1}$ since $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$.

Next, we claim that H'_w is isomorphic to P_4 . Observe that $v_{-(d+3)} \sim v_{d+1}$ since $|(d+1) - (-(d+3))| = 2d+4 \equiv -4 \pmod{2d+8} \in \langle S \rangle$. Next, we note that $v_{-(d+3)} \sim v_{d+2}$ since $|(d+2) - (-(d+3))| = 2d+5 \equiv -3 \pmod{2d+8} \in \langle S \rangle$, and $v_{d+4} \sim v_{d+1}$ since $|(d+4) - (d+1)| = 3 \in S$. We also note that v_{d+2} is adjacent to neither v_{d+1} nor v_{d+4} since $|(d+2) - (d+1)| = 1 \notin S$ and $|(d+4) - (d+2)| = 2 \notin S$. Furthermore, $v_{-(d+3)} \not\sim v_{d+4}$ since $|(d+4) - (-(d+3))| = 2d+7 \equiv -1 \pmod{2d+8} \notin \langle S \rangle$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Furthermore, H''_w and H'''_w are well-covered with $\beta(H''_w) = \beta(H'''_w) = 1$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Hence, G is well-covered and $\beta(G) = 6$, concluding the proof of Case 5.5.14.

Case 5.5.15 $d \geq 7$ and $n = 2d + 9$.

In this case $V(H) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_H[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $w \in N_H[v_1]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.5.15.1 $w = v_1$.

In this case $V(H_w) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_2, v_{d+2}, v_{d+3}, v_{d+4}\}$.

To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_2$.

Since v_2 is adjacent to v_{2+j} and v_{2-j} for each j in the set S , it follows that

$N_{H_w}[v_2] = \{v_{-1}, v_2, v_{d+2}\}$. We will show for each $u \in N_{H_w}[v_2]$ that $H_u =$

$H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 3$.

Case 5.5.15.1.1 $u = v_2$.

In this case $V(H_u) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+3}, v_{d+4}\}$. To show that

H_u is well-covered we are going to apply Lemma 5.1 with $x_0 = v_{-(d+3)}$. Since

$v_{-(d+3)}$ is adjacent to $v_{-(d+3)+j}$ and $v_{-(d+3)-j}$ for each j in the set S , it follows

that $N_{H_u}[v_{-(d+3)}] = \{v_{-(d+3)}, v_{d+3}\}$. We will show for each $x \in N_{H_u}[v_{-(d+3)}]$

that $H_x = H_u \setminus N_{H_u}[x]$ is well-covered with $\beta(H_x) = 2$.

Case 5.5.15.1.1.1 $x = v_{-(d+3)}$.

In this case $V(H_x) = \{v_{-(d+4)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+4}\}$. Now we show that H_x is

isomorphic to P_4 . Observe that v_{d+4} is adjacent to both $v_{-(d+2)}$ and $v_{-(d+1)}$ since

$|(d+4) - (-(d+2))| = 2d+6 \equiv -3 \pmod{2d+9} \in \langle S \rangle$ and $|(d+4) - (-(d+1))| =$

$2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$. Next, we note that $v_{-(d+1)} \sim v_{-(d+4)}$ since

$|-(d+1) - (-(d+4))| = 3 \in S$, and $v_{-(d+2)}$ is adjacent to neither $v_{-(d+4)}$ nor

$v_{-(d+1)}$ since $|-(d+2) - (-(d+4))| = 2 \notin S$ and $|-(d+1) - (-(d+2))| =$

$1 \notin S$. Furthermore, $v_{d+4} \not\sim v_{-(d+4)}$ since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1$

$\pmod{2d+9} \notin \langle S \rangle$. Hence, H_x is isomorphic to P_4 , and thus is well-covered

with $\beta(H_x) = 2$.

Case 5.5.15.1.1.2 $x = v_{d+3}$.

In this case $V(H_x) = \{v_{-(d+4)}, v_{d+4}\}$. Note that $v_{-(d+4)} \not\sim v_{d+4}$ since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \notin \langle S \rangle$. Hence, H_x is well-covered with $\beta(H_x) = 2$.

This concludes Case 5.5.15.1.1.

Case 5.5.15.1.2 $u = v_{d+2}$.

Let $H'_u = H_w[\{v_{-1}\}]$, $H''_u = H_w[\{v_{d+3}\}]$ and $H'''_u = H_w[\{v_{d+4}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ and $V(H'''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in one of the graphs H'_u , H''_u and H'''_u is adjacent to a vertex in either of the other two graphs. Observe that v_{d+3} is adjacent to neither v_{-1} nor v_{d+4} since $|(d+3) - (-1)| = d+4 \notin S$ and $|(d+4) - (d+3)| = 1 \notin S$.

We now note that $v_{d+4} \not\sim v_{-1}$ since $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Hence, H'_u , H''_u and H'''_u are well-covered with $\beta(H'_u) = \beta(H''_u) = \beta(H'''_u) = 1$. Therefore, H_u is well-covered with $\beta(H_u) = 3$.

Case 5.5.15.1.3 $u = v_{-1}$.

In this case $V(H_u) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{d+2}, v_{d+3}, v_{d+4}\}$. To show that H_u is well-covered we are going to apply Lemma 5.1 with $y_0 = v_{-(d+4)}$. Since $v_{-(d+4)}$ is adjacent to $v_{-(d+4)+j}$ and $v_{-(d+4)-j}$ for each j in the set S , it follows that $N_{H_u}[v_{-(d+4)}] = \{v_{-(d+4)}, v_{d+2}\}$. We will show for each $y \in N_{H_u}[v_{-(d+4)}]$ that $H_y = H_u \setminus N_{H_u}[y]$ is well-covered with $\beta(H_y) = 2$.

Case 5.5.15.1.3.1 $y = v_{-(d+4)}$.

In this case $V(H_y) = \{v_{-(d+3)}, v_{-(d+2)}, v_{d+3}, v_{d+4}\}$. We claim that H_y is isomorphic to P_4 . Observe that $v_{-(d+2)}$ is adjacent to both v_{d+3} and v_{d+4} since

$|(d+3) - (-(d+2))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$ and $|(d+4) - (-(d+2))| = 2d+6 \equiv -3 \pmod{2d+9} \in \langle S \rangle$. Next, we note that $v_{-(d+3)} \sim v_{d+3}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -3 \pmod{2d+9} \in \langle S \rangle$. Also note that $v_{-(d+3)} \not\sim v_{-(d+2)}$ since $|(-(d+2)) - (-(d+3))| = 1 \notin S$, and $v_{d+3} \not\sim v_{d+4}$ since $|(d+4) - (d+3)| = 1 \notin S$. Furthermore, $v_{d+4} \not\sim v_{-(d+3)}$ since $|(d+4) - (-(d+3))| = 2d+7 \equiv -2 \pmod{2d+9} \notin \langle S \rangle$. Hence, H_y is isomorphic to P_4 , and thus is well-covered with $\beta(H_y) = 2$.

Case 5.5.15.1.3.2 $y = v_{d+2}$.

In this case $V(H_y) = \{v_{d+3}, v_{d+4}\}$. Note that $v_{d+4} \not\sim v_{d+3}$ since $|(d+4) - (d+3)| = 1 \notin S$. Hence, H_y is well-covered with $\beta(H_y) = 2$.

This concludes Case 5.5.15.1.3 and Case 5.5.15.1.

Case 5.5.15.2 $w = v_{d+1}$.

In this case $V(H_w) = \{v_{-2}, v_{-1}, v_{d+2}, v_{d+3}\}$. From Case 5.5.11, we know that H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.15.3 $w = v_{-2}$.

Let $H'_w = H[\{v_{-(d+4)}, v_{-(d+3)}, v_{d+1}, v_{d+2}, v_{d+3}, v_{d+4}\}]$ and $H''_w = H[\{v_{-1}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. Observe that v_{-1} is adjacent to neither $v_{-(d+4)}$, $v_{-(d+3)}$, v_{d+1} , v_{d+2} , v_{d+3} nor v_{d+4} since $|(-1) - (-(d+4))| = d+3 \notin \langle S \rangle$, $|(-1) - (-(d+3))| = d+2 \notin \langle S \rangle$, $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$, $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$ and $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Hence, no vertex in H'_w is adjacent to a vertex in H''_w .

Next, to show that H'_w is well-covered we are going to apply Lemma 5.1 with $z_0 = v_{d+3}$. Since v_{d+3} is adjacent to $v_{(d+3)+j}$ and $v_{(d+3)-j}$ for each j in the set S , it follows that $N_{H'_w}[v_{d+3}] = \{v_{-(d+3)}, v_{d+3}\}$. We will show for each $z \in N_{H'_w}[v_{d+3}]$ that $H_z = H'_w \setminus N_{H'_w}[z]$ is well-covered with $\beta(H_z) = 2$.

Case 5.5.15.3.1 $z = v_{d+3}$.

In this case $V(H_z) = \{v_{-(d+4)}, v_{d+1}, v_{d+2}, v_{d+4}\}$. We claim that H_z is isomorphic to P_4 . Observe that $v_{d+2} \sim v_{-(d+4)}$ since $|(d+2) - (-(d+4))| = 2d+6 \equiv -3 \pmod{2d+9} \in \langle S \rangle$. Next, we note that v_{d+1} is adjacent to both $v_{-(d+4)}$ and v_{d+4} since $|(d+1) - (-(d+4))| = 2d+5 \equiv -4 \pmod{2d+9} \in \langle S \rangle$ and $|(d+4) - (d+1)| = 3 \in S$. Also note that v_{d+4} is adjacent to neither $v_{-(d+4)}$ nor v_{d+2} since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \notin \langle S \rangle$ and $|(d+4) - (d+2)| = 2 \notin S$. Furthermore, $v_{d+2} \not\sim v_{d+1}$ since $|(d+2) - (d+1)| = 1 \notin S$. Hence, H_z is isomorphic to P_4 , and thus is well-covered with $\beta(H_z) = 2$.

Case 5.5.15.3.2 $z = v_{-(d+3)}$.

In this case $V(H_z) = \{v_{-(d+4)}, v_{d+4}\}$. Note that $v_{-(d+4)} \not\sim v_{d+4}$ since $|(d+4) - (-(d+4))| = 2d+8 \equiv -1 \pmod{2d+9} \notin \langle S \rangle$. Hence, H_z is well-covered with $\beta(H_z) = 2$.

Finally, we note that H''_w is well-covered with $\beta(H''_w) = 1$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

This concludes Case 5.5.15.3.

Hence, G is well-covered and $\beta(G) = 6$, concluding the proof of Case 5.5.15.

Case 5.5.16 $d \geq 8$ and $2d + 10 \leq n \leq 3d + 2$.

To show that H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{\lfloor \frac{n}{2} \rfloor}$. Since $v_{\lfloor \frac{n}{2} \rfloor}$ is adjacent to $v_{\lfloor \frac{n}{2} \rfloor + j}$ and $v_{\lfloor \frac{n}{2} \rfloor - j}$ for each j in the set S , it follows that $N_H[v_{\lfloor \frac{n}{2} \rfloor}] = \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 3\} \cup \{v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_i : \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n - (d+1)\}$. We will show for each $w \in N_H[v_{\lfloor \frac{n}{2} \rfloor}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.5.16.1 $w = v_{\lfloor \frac{n}{2} \rfloor}$.

Let $H'_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_w = H[\{v_{\lfloor \frac{n}{2} \rfloor - 2}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. We also note that no vertex in H'_w is adjacent to a vertex in H''_w .

We claim that H'_w is isomorphic to P_4 . Observe that v_{-2} is adjacent to both v_{-1} and v_2 since $|1 - (-2)| = 3 \in S$ and $|2 - (-2)| = 4 \in S$. Next, we note that v_{-1} is adjacent to neither v_{-1} nor v_2 since $|1 - (-1)| = 2 \notin S$ and $|2 - 1| = 1 \notin S$. By symmetry, we can also deduce that $v_2 \sim v_{-1}$ and $v_{-1} \not\sim v_{-2}$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Next, we show that H''_w is isomorphic to P_4 . Observe that $v_{\lfloor \frac{n}{2} \rfloor + 2}$ is adjacent to both $v_{\lfloor \frac{n}{2} \rfloor - 2}$ and $v_{\lfloor \frac{n}{2} \rfloor - 1}$ since $|(\lfloor \frac{n}{2} \rfloor + 2) - (\lfloor \frac{n}{2} \rfloor - 2)| = 4 \in S$ and $|(\lfloor \frac{n}{2} \rfloor + 2) - (\lfloor \frac{n}{2} \rfloor - 1)| = 3 \in S$. Next, we note that $v_{\lfloor \frac{n}{2} \rfloor - 2} \sim v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|(\lfloor \frac{n}{2} \rfloor + 1) - (\lfloor \frac{n}{2} \rfloor - 2)| = 3 \in S$. Also note that $v_{\lfloor \frac{n}{2} \rfloor - 1}$ is adjacent to neither $v_{\lfloor \frac{n}{2} \rfloor - 2}$ nor $v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|(\lfloor \frac{n}{2} \rfloor - 1) - (\lfloor \frac{n}{2} \rfloor - 2)| = 1 \notin S$ and $|(\lfloor \frac{n}{2} \rfloor + 1) - (\lfloor \frac{n}{2} \rfloor - 1)| = 2 \notin S$. Furthermore, $v_{\lfloor \frac{n}{2} \rfloor + 2} \not\sim v_{\lfloor \frac{n}{2} \rfloor + 1}$ since $|(\lfloor \frac{n}{2} \rfloor + 2) - (\lfloor \frac{n}{2} \rfloor + 1)| = 1 \notin S$. Hence, H''_w is isomorphic to P_4 , and thus is well-covered with $\beta(H''_w) = 2$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.16.2 $w = v_{d+1}$ or $w = v_{-(d+1)}$.

By symmetry, we need only examine $w = v_{d+1}$. In this case $V(H_w) = \{v_{-2}, v_{-1}, v_{d+2}, v_{d+3}\}$. From Case 5.5.11, we know that H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.16.3 $w = v_{d+2}$ or $w = v_{-(d+2)}$.

By symmetry, we need only examine $w = v_{d+2}$. Let $H'_w = H[\{v_{-2}, v_1, v_{d+1}, v_{d+4}\}]$, $H''_w = H[\{v_{d+3}\}]$ and $H'''_w = H[\{v_{-1}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ and $V(H'''_w)$ forms a partition of $V(H_w)$.

We claim that no vertex in one of the graphs H'_w , H''_w and H'''_w is adjacent to a vertex in either of the other two graphs. Observe that v_{d+3} is adjacent to neither v_{-2}, v_1, v_{d+1} nor v_{d+4} since $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$, $|(d+3) - 1| = d+2 \notin \langle S \rangle$, $|(d+3) - (d+1)| = 2 \notin S$ and $|(d+3) - (d+4)| = 1 \notin S$. Next, we note that v_{-1} is adjacent to neither v_{-2}, v_1, v_{d+1} nor v_{d+4} since $|(-1) - (-2)| = 1 \notin S$, $|1 - (-1)| = 2 \notin S$, $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$ and $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$. Furthermore, $v_{d+3} \not\sim v_{-1}$ since $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$.

Next, we show that H'_w is isomorphic to P_4 . Observe that v_1 is adjacent to both v_{-2} and v_{d+1} since $|1 - (-2)| = 3 \in S$ and $|(d+1) - 1| = d \in S$. Next, we note that v_{d+4} is adjacent to neither v_{-2} nor v_1 since $|(d+4) - (-2)| = d+6 \notin \langle S \rangle$ and $|(d+4) - 1| = d+3 \notin \langle S \rangle$. Furthermore, $v_{d+1} \not\sim v_{-2}$ since $|(d+1) - (-2)| = d+3 \notin \langle S \rangle$, and $v_{d+1} \sim v_{d+4}$ since $|(d+4) - (d+1)| = 3 \in S$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

Finally, we note that H''_w and H'''_w are well-covered with $\beta(H''_w) = \beta(H'''_w) = 1$. Therefore, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.16.4 $w = v_{d+3}$ or $w = v_{-(d+3)}$ and $d \geq 10$. Note that this is only valid if $2d + 12 \leq n \leq 3d + 2$.

By symmetry, we need only examine $w = v_{d+3}$. In this case $V(H_w) = \{v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+4}, v_{d+5}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{-2}$. Since v_{-2} is adjacent to v_{-2+j} and v_{-2-j} for each j in the set S , it follows that $N_{H_w}[v_{-2}] = \{v_{-2}, v_1, v_2\}$. We will show for each $u \in N_{H_w}[v_{-2}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 3$.

Case 5.5.16.4.1 $u = v_{-2}$.

Let $H'_u = H_w[\{v_{d+1}, v_{d+2}, v_{d+4}, v_{d+5}\}]$ and $H''_u = H_w[\{v_{-1}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_w)$. Observe that v_{-1} is adjacent to neither v_{d+1} , v_{d+2} , v_{d+4} nor v_{d+5} since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$, $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$ and $|(d+5) - (-1)| = d+6 \notin \langle S \rangle$. Hence, no vertex in H'_u is adjacent to a vertex in H''_u .

Next, we show that H'_u is isomorphic to P_4 . Observe that v_{d+1} is adjacent to both v_{d+4} and v_{d+5} since $|(d+4) - (d+1)| = 3 \in S$ and $|(d+5) - (d+1)| = 4 \in S$. Next, we note that $v_{d+2} \sim v_{d+5}$ since $|(d+5) - (d+2)| = 3 \in S$ and $v_{d+1} \not\sim v_{d+2}$ since $|(d+2) - (d+1)| = 1 \notin S$. Furthermore, v_{d+4} is adjacent to neither v_{d+2} nor v_{d+5} since $|(d+4) - (d+2)| = 2 \notin S$ and $|(d+5) - (d+4)| = 1 \notin S$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that H''_u is well-covered with $\beta(H''_u) = 1$, and thus H_u is well-covered with $\beta(H_u) = 3$.

Case 5.5.16.4.2 $u = v_1$.

Let $H'_u = H_w[\{v_{-1}, v_2, v_{d+2}, v_{d+5}\}]$ and $H''_u = H_w[\{v_{d+4}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_w)$. Observe that v_{d+4} is adjacent to neither v_{-1} , v_2 , v_{d+2} nor v_{d+5} since $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$, $|(d+4) - 2| = d+2 \notin \langle S \rangle$, $|(d+4) - (d+2)| = 2 \notin S$ and $|(d+5) - (d+4)| = 1 \notin S$. Hence, no vertex in H'_u is adjacent to a vertex in H''_u .

Next, we show that H'_u is isomorphic to P_4 . Observe that v_{d+2} is adjacent to both v_2 and v_{d+5} since $|(d+2) - 2| = d \in S$ and $|(d+5) - (d+2)| = 3 \in S$. Next, we note that $v_2 \sim v_{-1}$ since $|2 - (-1)| = 3 \in S$, and $v_{d+2} \not\sim v_{-1}$ since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Furthermore, v_{d+5} is adjacent to neither v_{-1} nor v_2 since $|(d+5) - (-1)| = d+6 \notin \langle S \rangle$ and $|(d+5) - 2| = d+3 \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that H''_u is well-covered with $\beta(H''_u) = 1$, and hence H_u is well-covered with $\beta(H_u) = 3$.

Case 5.5.16.4.3 $u = v_2$.

In this case $V(H_u) = \{v_1, v_{d+4}, v_{d+5}\}$. Observe that v_1 is adjacent to neither v_{d+4} nor v_{d+5} since $|(d+4) - 1| = d+3 \notin \langle S \rangle$ and $|(d+5) - 1| = d+4 \notin \langle S \rangle$. Furthermore, $v_{d+4} \not\sim v_{d+5}$ since $|(d+5) - (d+4)| = 1 \notin S$. Hence, H_u is well-covered with $\beta(H_u) = 3$.

This concludes Case 5.5.16.4.

Case 5.5.16.5 $w = v_{2-\lfloor \frac{n}{2} \rfloor}$. *Note that this is only valid if n is odd.*

Let $H'_w = H[\{v_{-\lfloor \frac{n}{2} \rfloor}, v_{1-\lfloor \frac{n}{2} \rfloor}, v_{3-\lfloor \frac{n}{2} \rfloor}, v_{4-\lfloor \frac{n}{2} \rfloor}\}]$ and $H''_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. We also note that no vertex in H'_w is adjacent to a vertex in H''_w .

Next, we show that H'_w is isomorphic to P_4 . Observe that $v_{4-\lfloor \frac{n}{2} \rfloor}$ is adjacent to both $v_{-\lfloor \frac{n}{2} \rfloor}$ and $v_{1-\lfloor \frac{n}{2} \rfloor}$ since $|(4 - \lfloor \frac{n}{2} \rfloor) - (-\lfloor \frac{n}{2} \rfloor)| = 4 \in S$, and $|(4 - \lfloor \frac{n}{2} \rfloor) - (1 - \lfloor \frac{n}{2} \rfloor)| = 3 \in S$. Next, we note that $v_{1-\lfloor \frac{n}{2} \rfloor}$ is adjacent to neither $v_{-\lfloor \frac{n}{2} \rfloor}$ nor $v_{3-\lfloor \frac{n}{2} \rfloor}$ since $|(1 - \lfloor \frac{n}{2} \rfloor) - (-\lfloor \frac{n}{2} \rfloor)| = 1 \notin S$ and $|(3 - \lfloor \frac{n}{2} \rfloor) - (1 - \lfloor \frac{n}{2} \rfloor)| = 2 \notin S$. Furthermore, $v_{4-\lfloor \frac{n}{2} \rfloor} \not\sim v_{3-\lfloor \frac{n}{2} \rfloor}$ since $|(4 - \lfloor \frac{n}{2} \rfloor) - (3 - \lfloor \frac{n}{2} \rfloor)| = 1 \notin S$, and

$v_{3-\lfloor \frac{n}{2} \rfloor} \sim v_{-\lfloor \frac{n}{2} \rfloor}$ since $|(3 - \lfloor \frac{n}{2} \rfloor) - (-\lfloor \frac{n}{2} \rfloor)| = 3 \in S$. Hence, H'_w is isomorphic to P_4 , and thus is well-covered with $\beta(H'_w) = 2$.

From Case 5.5.16.1, we know that H''_w is well-covered with $\beta(H''_w) = 2$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.16.6 $w = v_k$ or $w = v_{-k}$ for $d + 4 \leq k \leq \lfloor \frac{n}{2} \rfloor - 3$ and $d \geq 12$.

By symmetry, we need only examine $w = v_k$. Since neither 1 nor 2 is in S , we can deduce that v_k is adjacent to neither v_{k-2} , v_{k-1} , v_{k+1} nor v_{k+2} . Let $H'_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_w = H[\{v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. We also note that no vertex in H'_w is adjacent to a vertex in H''_w .

Next, we show that H''_w is isomorphic to P_4 . Observe that v_{k+2} is adjacent to both v_{k-2} and v_{k-1} since $|(k+2) - (k-2)| = 4 \in S$ and $|(k+2) - (k-1)| = 3 \in S$. Next, we note that $v_{k+1} \sim v_{k-2}$ since $|(k+1) - (k-2)| = 3 \in S$, and $v_{k-2} \not\sim v_{k-1}$ since $|(k-1) - (k-2)| = 1 \notin S$. Furthermore, v_{k+1} is adjacent to neither v_{k-1} nor v_{k+2} since $|(k+1) - (k-1)| = 2 \notin S$ and $|(k+2) - (k+1)| = 1 \notin S$. Hence, H''_w is isomorphic to P_4 , and thus is well-covered with $\beta(H''_w) = 2$.

From Case 5.5.16.1, we know that H'_w is well-covered with $\beta(H'_w) = 2$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Hence, G is well-covered and $\beta(G) = 6$, concluding the proof of Case 5.5.16.

(vii) $d \geq 3$ and $n = 3d + 6$.

First, when $d = 3$ we note that $C(15, \{3\})$ is well-covered and $\beta(G) = 6$.

Next, we consider the case where $d \geq 4$. Then $V(H) = \{v_i : -\lfloor \frac{n}{2} \rfloor \leq i \leq -(d+1)\} \cup \{v_{-2}, v_{-1}\} \cup \{v_1, v_2\} \cup \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. To show that

H is well-covered we are going to apply Lemma 5.1 with $w_0 = v_{\lfloor \frac{n}{2} \rfloor}$. Since $v_{\lfloor \frac{n}{2} \rfloor}$ is adjacent to $v_{\lfloor \frac{n}{2} \rfloor + j}$ and $v_{\lfloor \frac{n}{2} \rfloor - j}$ for each j in the set S , it follows that $N_H[v_{\lfloor \frac{n}{2} \rfloor}] = \{v_i : d+1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 3\} \cup \{v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{v_i : \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n - (d+1)\}$. We will show for each $w \in N_H[v_{\lfloor \frac{n}{2} \rfloor}]$ that $H_w = H \setminus N_H[w]$ is well-covered with $\beta(H_w) = 4$.

Case 5.5.17 $w = v_{\lfloor \frac{n}{2} \rfloor}$.

Let $H'_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_w = H[\{v_{\lfloor \frac{n}{2} \rfloor - 2}, v_{\lfloor \frac{n}{2} \rfloor - 1}, v_{\lfloor \frac{n}{2} \rfloor + 1}, v_{\lfloor \frac{n}{2} \rfloor + 2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. From Case 5.5.16.1, we know that H'_w and H''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = 2$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.17.1 $w = v_{d+1}$ or $w = v_{-(d+1)}$.

By symmetry, we need only examine $w = v_{d+1}$. In this case $V(H_w) = \{v_{-(d+4)}, v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_{d+2}, v_{d+3}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $u_0 = v_{-1}$. Since v_{-1} is adjacent to v_{-1+j} and v_{-1-j} for each j in the set S , it follows that $N_{H_w}[v_{-1}] = \{v_{-(d+1)}, v_{-1}\}$. We will show for each $u \in N_{H_w}[v_{-1}]$ that $H_u = H_w \setminus N_{H_w}[u]$ is well-covered with $\beta(H_u) = 3$.

Case 5.5.17.1.1 $u = v_{-1}$.

Let $H'_u = H_w[\{v_{-(d+4)}, v_{-(d+3)}, v_{d+2}, v_{d+3}\}]$ and $H''_u = H_w[\{v_{-(d+2)}, v_{-2}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in H'_u is adjacent to a vertex in H''_u . Observe that v_{-2} is adjacent to neither $v_{-(d+4)}$, $v_{-(d+3)}$, v_{d+2} nor v_{d+3} since $|(-2) - (-(d+4))| = d+2 \notin \langle S \rangle$, $|(-2) - (-(d+3))| = d+1 \notin \langle S \rangle$, $|(d+2) - (-2)| = d+4 \notin \langle S \rangle$

and $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$. Next, we note that $v_{-(d+2)}$ is adjacent to neither $v_{-(d+4)}$ nor $v_{-(d+3)}$ since $|-(d+2) - (-(d+4))| = 2 \notin S$ and $|-(d+2) - (-(d+3))| = 1 \notin S$. Furthermore, $v_{-(d+2)}$ is adjacent to neither v_{d+2} nor v_{d+3} since $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+2) \pmod{3d+6} \notin \langle S \rangle$ and $|(d+3) - (-(d+2))| = 2d+5 \equiv -(d+1) \pmod{3d+6} \notin \langle S \rangle$.

Next, we claim that H'_u is isomorphic to P_4 . Observe that $v_{d+2} \sim v_{-(d+4)}$ since $|(d+2) - (-(d+4))| = 2d+6 \equiv -d \pmod{3d+6} \in \langle S \rangle$. Next, we note that $v_{d+3} \sim v_{-(d+4)}$ since $|(d+3) - (-(d+4))| = 2d+7 \equiv -(d-1) \pmod{3d+6} \in \langle S \rangle$. Also note that $v_{-(d+3)} \not\sim v_{-(d+4)}$ since $|-(d+3) - (-(d+4))| = 1 \notin S$, and $v_{d+2} \not\sim v_{d+3}$ since $|(d+3) - (d+2)| = 1 \notin S$. Furthermore, $v_{d+3} \sim v_{-(d+3)}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -d \pmod{3d+6} \in \langle S \rangle$, and $v_{d+2} \not\sim v_{-(d+3)}$ since $|(d+2) - (-(d+3))| = 2d+5 \equiv -(d+1) \pmod{3d+6} \notin \langle S \rangle$. Hence, H'_u is isomorphic to P_4 , and thus is well-covered with $\beta(H'_u) = 2$.

Finally, we note that $v_{-2} \sim v_{-(d+2)}$ since $|(-2) - (-(d+2))| = d \in S$, and thus H''_u is well-covered with $\beta(H''_u) = 1$. Hence, H_3 is well-covered with $\beta(H_u) = 3$.

Case 5.5.17.1.2 $u = v_{-(d+1)}$.

Let $H'_u = H_w[\{v_{-(d+3)}, v_{d+3}\}]$, $H''_u = H_w[\{v_{d+2}\}]$ and $H'''_u = H_w[\{v_{-(d+2)}\}]$. Note that $V(H'_u)$ together with $V(H''_u)$ and $V(H'''_u)$ forms a partition of $V(H_u)$.

We claim that no vertex in one of the graphs H'_u , H''_u and H'''_u is adjacent to a vertex in either of the other two graphs. First, we note that $v_{d+2} \not\sim v_{-(d+2)}$ since $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+2) \pmod{3d+6} \notin \langle S \rangle$, and $v_{d+2} \not\sim v_{-(d+3)}$ since $|(d+2) - (-(d+3))| = 2d+5 \equiv -(d+1) \pmod{3d+6} \notin \langle S \rangle$. Next, observe that $v_{d+2} \not\sim v_{d+3}$ since $|(d+3) - (d+2)| = 1 \notin S$. By symmetry, we can also deduce that $v_{-(d+2)} \not\sim v_{-(d+3)}$ and $v_{d+3} \not\sim v_{-(d+2)}$.

We now note that $v_{d+3} \sim v_{-(d+3)}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -d$

$(\text{mod } 3d + 6) \in \langle S \rangle$, and thus H'_u is well-covered with $\beta(H'_u) = 1$. Furthermore, H''_u and H'''_u are well-covered with $\beta(H''_u) = \beta(H'''_u) = 1$. Hence, H_u is well-covered with $\beta(H_u) = 3$.

This concludes Case 5.5.17.1.

Case 5.5.17.2 $w = v_{d+2}$ or $w = v_{-(d+2)}$.

By symmetry, we need only examine $w = v_{d+2}$. In this case $V(H_w) = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_{d+1}, v_{d+3}, v_{d+4}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $x_0 = v_{-(d+2)}$. Since $v_{-(d+2)}$ is adjacent to $v_{-(d+2)+j}$ and $v_{-(d+2)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+2)}] = \{v_{-2}, v_{-(d+2)}, v_{d+4}\}$. We will show for each $x \in N_{H_w}[v_{-(d+2)}]$ that $H_x = H_w \setminus N_{H_w}[x]$ is well-covered with $\beta(H_x) = 3$.

Case 5.5.17.2.1 $x = v_{-(d+2)}$.

Let $H'_x = H_w[\{v_1, v_{d+1}\}]$, $H''_x = H_w[\{v_{-1}, v_{-(d+1)}\}]$ and $H'''_x = H_w[\{v_{d+3}, v_{-(d+3)}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ and $V(H'''_x)$ forms a partition of $V(H_x)$.

We claim that no vertex in one of the graphs H'_x , H''_x and H'''_x is adjacent to a vertex in either of the other two graphs. Observe that v_1 is adjacent to neither $v_{-(d+3)}, v_{-(d+1)}, v_{-1}$ nor v_{d+3} since $|1 - (-(d+3))| = d+4 \notin \langle S \rangle$, $|1 - (-(d+1))| = d+2 \notin \langle S \rangle$, $|1 - (-1)| = 2 \notin S$, and $|(d+3) - 1| = d+2 \notin \langle S \rangle$. Next, we note that $v_{d+1} \not\sim v_{d+3}$ since $|(d+3) - (d+1)| = 2 \notin S$, and v_{d+1} is adjacent to neither $v_{-(d+3)}$ nor $v_{-(d+1)}$ since $|(d+1) - (-(d+3))| = 2d+4 \equiv -(d+2) \pmod{3d+6} \notin \langle S \rangle$ and $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+4) \pmod{3d+6} \notin \langle S \rangle$. By symmetry, we can also deduce that v_{-1} is adjacent to neither $v_{-(d+3)}, v_{d+1}$ nor v_{d+3} ; and $v_{-(d+1)}$ is adjacent to neither $v_{-(d+3)}$ nor v_{d+3} .

Next, we note that $v_1 \sim v_{d+1}$ since $|(d+1) - (1)| = d \in S$. Hence, H'_x is well-covered with $\beta(H'_x) = 1$. Similarly, H''_x is well-covered with $\beta(H''_x) = 1$. Furthermore, $v_{d+3} \sim v_{-(d+3)}$ since $|(d+3) - (-(d+3))| = 2d+6 \equiv -d \pmod{3d+6} \in \langle S \rangle$, and thus H'''_x is well-covered with $\beta(H'''_x) = 1$. Therefore, H_x is well-covered with $\beta(H_x) = 3$.

Case 5.5.17.2.2 $x = v_{d+4}$.

Let $H'_x = H_w[\{v_{-(d+1)}, v_{-2}, v_{-1}, v_1\}]$ and $H''_x = H_w[\{v_{d+3}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ forms a partition of $V(H_x)$. Note that v_{d+3} is adjacent to neither $v_{-(d+1)}$, v_{-2} , v_{-1} nor v_1 since $|(d+3) - (-(d+1))| = 2d+4 \equiv -(d+2) \pmod{3d+6} \notin \langle S \rangle$, $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$, $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$ and $|(d+3) - 1| = d+2 \notin \langle S \rangle$. Hence, no vertex in H'_x is adjacent to a vertex in H''_x .

Next, we show that H'_x is isomorphic to P_4 . Observe that $v_{-(d+1)}$ is adjacent to both v_{-2} and v_{-1} since $|(-2) - (-(d+1))| = d-1 \in S$ and $|(-1) - (-(d+1))| = d \in S$. Next, we note that $v_{-2} \sim v_1$ since $|1 - (-2)| = 3 \in S$, and $v_{-(d+1)} \not\sim v_1$ since $|1 - (-(d+1))| = d+2 \notin \langle S \rangle$. Furthermore, v_{-1} is adjacent to neither v_{-2} nor v_1 since $|(-1) - (-2)| = 1 \notin S$ and $|1 - (-1)| = 2 \notin S$. Hence, H'_x is isomorphic to P_4 , and thus is well-covered with $\beta(H'_x) = 2$.

Finally, we note that H''_x is well-covered with $\beta(H''_x) = 1$. Therefore, H_x is well-covered with $\beta(H_x) = 3$.

Case 5.5.17.2.3 $x = v_{-2}$.

Let $H'_x = H_w[\{v_{-(d+3)}, v_{d+1}, v_{d+3}, v_{d+4}\}]$ and $H''_x = H_w[\{v_{-1}\}]$. Note that $V(H'_x)$ together with $V(H''_x)$ forms a partition of $V(H_x)$. Note that v_{-1} is adjacent to neither $v_{-(d+3)}$, v_{d+1} , v_{d+3} nor v_{d+4} since $|(-1) - (-(d+3))| =$

$d + 2 \notin \langle S \rangle$, $|(d + 1) - (-1)| = d + 2 \notin \langle S \rangle$, $|(d + 3) - (-1)| = d + 4 \notin \langle S \rangle$, and $|(d + 4) - (-1)| = d + 5 \notin \langle S \rangle$. Hence, no vertex in H'_x is adjacent to a vertex in H''_x .

Next, we show that H'_x is isomorphic to P_4 . Observe that $v_{-(d+3)}$ is adjacent to both v_{d+3} and v_{d+4} since $|(d + 3) - (-(d + 3))| = 2d + 6 \equiv -d \pmod{3d + 6} \in \langle S \rangle$ and $|(d + 4) - (-(d + 3))| = 2d + 7 \equiv -(d - 1) \pmod{3d + 6} \in \langle S \rangle$. Next, we note that $v_{d+4} \sim v_{d+1}$ since $|(d + 4) - (d + 1)| = 3 \in S$, and $v_{d+1} \not\sim v_{-(d+3)}$ since $|(d + 1) - (-(d + 3))| = 2d + 4 \equiv -(d + 2) \pmod{3d + 6} \notin \langle S \rangle$. Furthermore, v_{d+3} is adjacent to neither v_{d+1} nor v_{d+4} since $|(d + 3) - (d + 1)| = 2 \notin S$ and $|(d + 4) - (d + 3)| = 1 \notin S$. Hence, H'_x is isomorphic to P_4 , and thus is well-covered with $\beta(H'_x) = 2$.

Finally, we note that H''_x is well-covered with $\beta(H''_x) = 1$. Hence, H_x is well-covered with $\beta(H_x) = 3$.

This concludes Case 5.5.17.2.

Case 5.5.17.3 $w = v_{d+3}$ or $w = v_{-(d+3)}$ and $d \geq 6$.

By symmetry, we need only examine $w = v_{d+3}$. In this case $V(H_w) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+1}, v_{d+2}, v_{d+4}, v_{d+5}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $z_0 = v_1$. Since v_1 is adjacent to v_{1+j} and v_{1-j} for each j in the set S , it follows that $N_{H_w}[v_1] = \{v_{-2}, v_1, v_{d+1}\}$. We will show for each $z \in N_{H_w}[v_{-(d+2)}]$ that $H_z = H_w \setminus N_{H_w}[z]$ is well-covered with $\beta(H_z) = 3$.

Case 5.5.17.3.1 $z = v_1$.

In this case $V(H_z) = \{v_{-(d+2)}, v_{-(d+1)}, v_{-1}, v_2, v_{d+2}, v_{d+4}, v_{d+5}\}$. To show that H_z is well-covered we are going to apply Lemma 5.1 with $q_0 = v_{d+4}$. Since

v_{d+4} is adjacent to $v_{(d+4)+j}$ and $v_{(d+4)-j}$ for each j in the set S , it follows that $N_{H_z}[v_{d+4}] = \{v_{-(d+2)}, v_{d+4}\}$. We will show for each $q \in N_{H_z}[v_{d+4}]$ that $H_q = H_w \setminus N_{H_w}[q]$ is well-covered with $\beta(H_q) = 2$.

Case 5.5.17.3.1.1 $q = v_{d+4}$.

In this case $V(H_q) = \{v_{-(d+1)}, v_{-1}, v_2, v_{d+2}, v_{d+5}\}$. We claim that H_q is isomorphic to C_5 . Observe that v_{d+2} is adjacent to both v_2 and v_{d+5} since $|(d+2)-2| = d \in S$ and $|(d+5)-(d+2)| = 3 \in S$. Next, we note that v_{-1} is adjacent to both v_2 and $v_{-(d+1)}$ since $|2-(-1)| = 3 \in S$ and $|(-1)-(-(d+1))| = d \in S$. We also note that v_{d+2} is adjacent to neither v_{-1} nor $v_{-(d+1)}$ since $|(d+2)-(-1)| = d+3 \notin \langle S \rangle$ and $|(d+2)-(-(d+1))| = 2d+3 \equiv -(d+3) \pmod{3d+6} \notin \langle S \rangle$; and v_{d+5} is adjacent to neither v_{-1} nor v_2 since $|(d+5)-(-1)| = d+6 \notin \langle S \rangle$ and $|(d+5)-2| = d+3 \notin \langle S \rangle$. Furthermore, $v_{-(d+1)} \sim v_{d+5}$ since $|(d+5)-(-(d+1))| = 2d+6 \equiv -d \pmod{3d+6} \in \langle S \rangle$, and $v_{-(d+1)} \not\sim v_2$ since $|2-(-(d+1))| = d+3 \notin \langle S \rangle$. Hence, H_q is isomorphic to C_5 , and thus is well-covered with $\beta(H_q) = 2$.

Case 5.5.17.3.1.2 $q = v_{-(d+2)}$.

In this case $V(H_q) = \{v_{-(d+1)}, v_{-1}, v_2, v_{d+2}\}$. From Case 5.5.17.3.1.1, we know that v_{d+2} is adjacent to v_2 ; v_{-1} is adjacent to both v_2 and $v_{-(d+1)}$; v_{d+2} is adjacent to neither v_{-1} nor $v_{-(d+1)}$; and $v_{-(d+1)} \not\sim v_2$. Hence, H_q is isomorphic to P_4 , and thus is well-covered with $\beta(H_q) = 2$.

This concludes Case 5.5.17.3.1.

Case 5.5.17.3.2 $z = v_{d+1}$.

Let $H'_z = H_w[\{v_{-(d+2)}, v_{-(d+1)}, v_{-2}, v_{-1}\}]$ and $H''_z = H_w[\{v_{d+2}\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ forms a partition of $V(H_z)$. Note that v_{d+2} is adjacent to neither $v_{-(d+2)}, v_{-(d+1)}, v_{-2}$ nor v_{-1} since $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+2) \pmod{3d+6} \notin \langle S \rangle$, $|(d+2) - (-(d+1))| = 2d+3 \equiv -(d+3) \pmod{3d+6} \notin \langle S \rangle$, $|(d+2) - (-2)| = d+4 \notin \langle S \rangle$ and $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$. Hence, no vertex in H'_z is adjacent to a vertex in H''_z .

We claim that H'_z is isomorphic to P_4 . Observe that $v_{-(d+1)}$ is adjacent to both v_{-2} and v_{-1} since $|(-2) - (-(d+1))| = d-1 \in S$ and $|(-1) - (-(d+1))| = d \in S$. Next, we note that $v_{-2} \sim v_{-(d+2)}$ since $|(-2) - (-(d+2))| = d \in S$, and $v_{-2} \not\sim v_{-1}$ since $|(-1) - (-2)| = 1 \notin S$. Furthermore, $v_{-(d+2)}$ is adjacent to neither $v_{-(d+1)}$ nor v_{-1} since $|-(d+1) - (-(d+2))| = 1 \notin S$ and $|(-1) - (-(d+2))| = d+1 \notin \langle S \rangle$. Hence, H'_z is isomorphic to P_4 , and thus is well-covered with $\beta(H'_z) = 2$.

Finally, we note that H''_z is well-covered with $\beta(H''_z) = 1$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

Case 5.5.17.3.3 $z = v_{-2}$.

Let $H'_z = H_w[\{v_{d+1}, v_{d+2}, v_{d+4}, v_{d+5}\}]$ and $H''_z = H_w[\{v_{-1}\}]$. Note that $V(H'_z)$ together with $V(H''_z)$ forms a partition of $V(H_z)$. Note that v_{-1} is adjacent to neither $v_{d+1}, v_{d+2}, v_{d+4}$ nor v_{d+5} since $|(d+1) - (-1)| = d+2 \notin \langle S \rangle$, $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$, $|(d+4) - (-1)| = d+5 \notin \langle S \rangle$ and $|(d+5) - (-1)| = d+6 \notin \langle S \rangle$. Hence, no vertex in H'_z is adjacent to a vertex in H''_z .

Next, we show that H'_z is isomorphic to P_4 . Observe that v_{d+1} is adjacent to both v_{d+4} and v_{d+5} since $|(d+4) - (d+1)| = 3 \in S$ and $|(d+5) - (d+1)| = 4 \in S$. Next, we note that $v_{d+2} \sim v_{d+5}$ since $|(d+5) - (d+2)| = 3 \in S$, and $v_{d+1} \not\sim v_{d+2}$ since $|(d+2) - (d+1)| = 1 \notin S$. Furthermore, v_{d+4} is adjacent to neither v_{d+2}

nor v_{d+5} since $|(d+4) - (d+2)| = 2 \notin S$ and $|(d+5) - (d+4)| = 1 \notin S$. Hence, H'_z is isomorphic to P_4 , and thus is well-covered with $\beta(H'_z) = 2$.

Finally, we note that H''_z is well-covered with $\beta(H''_z) = 1$. Therefore, H_z is well-covered with $\beta(H_z) = 3$.

This concludes Case 5.5.17.3.

Case 5.5.17.4 $w = v_{d+4}$ or $w = v_{-(d+4)}$ and $d \geq 8$.

By symmetry, we need only examine $w = v_{d+4}$. In this case $V(H_w) = \{v_{-(d+1)}, v_{-2}, v_{-1}, v_1, v_2, v_{d+2}, v_{d+3}, v_{d+5}, v_{d+6}\}$. To show that H_w is well-covered we are going to apply Lemma 5.1 with $r_0 = v_{-(d+1)}$. Since $v_{-(d+1)}$ is adjacent to $v_{-(d+1)+j}$ and $v_{-(d+1)-j}$ for each j in the set S , it follows that $N_{H_w}[v_{-(d+1)}] = \{v_{-(d+1)}, v_{-2}, v_{-1}, v_{d+5}, v_{d+6}\}$. We will show for each $r \in N_{H_w}[v_{-(d+1)}]$ that $H_r = H_w \setminus N_{H_w}[r]$ is well-covered with $\beta(H_r) = 3$.

Case 5.5.17.4.1 $r = v_{-(d+1)}$.

Let $H'_r = H_w[\{v_2, v_{d+2}\}]$, $H''_r = H_w[\{v_{d+3}\}]$ and $H'''_r = H_w[\{v_1\}]$. Note that $V(H'_r)$ together with $V(H''_r)$ and $V(H'''_r)$ forms a partition of $V(H_r)$.

We claim that no vertex in one of the graphs H'_r , H''_r and H'''_r is adjacent to a vertex in either of the other two graphs. Observe that v_{d+3} is adjacent to neither v_2 nor v_{d+2} since $|(d+3) - 2| = d+1 \notin \langle S \rangle$ and $|(d+3) - (d+2)| = 1 \notin S$. Next, we note that v_1 is adjacent to neither v_2 nor v_{d+2} since $|2 - 1| = 1 \notin S$ and $|(d+2) - 1| = d+1 \notin \langle S \rangle$. Furthermore, $v_1 \not\sim v_{d+3}$ since $|(d+3) - 1| = d+2 \notin \langle S \rangle$.

We now note that $v_{d+2} \sim v_2$ since $|(d+2) - 2| = d \in S$, and thus H'_r is well-covered with $\beta(H'_r) = 1$. Furthermore, H''_r and H'''_r are well-covered with $\beta(H''_r) = \beta(H'''_r) = 1$. Hence, H_r is well-covered with $\beta(H_r) = 3$.

Case 5.5.17.4.2 $r = v_{d+6}$.

Let $H'_r = H_w[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_r = H_w[\{v_{d+5}\}]$. Note that $V(H'_r)$ together with $V(H''_r)$ forms a partition of $V(H_r)$. Note that v_{d+5} is adjacent to neither v_{-2}, v_{-1}, v_1 nor v_2 since $|(d+5) - (-2)| = d+7 \notin \langle S \rangle$, $|(d+5) - (-1)| = d+6 \notin \langle S \rangle$, $|(d+5) - 1| = d+4 \notin \langle S \rangle$, and $|(d+5) - 2| = d+3 \notin \langle S \rangle$. Hence, no vertex in H'_r is adjacent to a vertex in H''_r .

From Case 5.5.16.1, we know that H'_r is well-covered with $\beta(H'_r) = 2$. Furthermore, H''_r is well-covered with $\beta(H''_r) = 1$. Hence, H_r is well-covered with $\beta(H_r) = 3$.

Case 5.5.17.4.3 $r = v_{-2}$.

Let $H'_r = H_w[\{v_{d+2}, v_{d+3}, v_{d+5}, v_{d+6}\}]$ and $H''_r = H_w[\{v_{-1}\}]$. Note that $V(H'_r)$ together with $V(H''_r)$ forms a partition of $V(H_r)$. Note that v_{-1} is adjacent to neither $v_{d+2}, v_{d+3}, v_{d+5}$ nor v_{d+6} since $|(d+2) - (-1)| = d+3 \notin \langle S \rangle$, $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$, $|(d+5) - (-1)| = d+6 \notin \langle S \rangle$ and $|(d+6) - (-1)| = d+7 \notin \langle S \rangle$. Hence, no vertex in H'_r is adjacent to a vertex in H''_r .

Next, we show that H'_r is isomorphic to P_4 . Observe that v_{d+6} is adjacent to both v_{d+2} and v_{d+3} since $|(d+6) - (d+2)| = 4 \in S$ and $|(d+6) - (d+3)| = 3 \in S$. Next, we note that $v_{d+2} \sim v_{d+5}$ since $|(d+5) - (d+2)| = 3 \in S$, and $v_{d+5} \not\sim v_{d+6}$ since $|(d+6) - (d+5)| = 1 \notin S$. Furthermore, v_{d+3} is adjacent to neither v_{d+2} nor v_{d+5} since $|(d+3) - (d+2)| = 1 \notin S$ and $|(d+5) - (d+3)| = 2 \notin S$. Hence, H'_r is isomorphic to P_4 , and thus is well-covered with $\beta(H'_r) = 2$.

Finally, we note that H''_r is well-covered with $\beta(H''_r) = 1$. Hence, H_r is well-covered with $\beta(H_r) = 3$.

Case 5.5.17.4.4 $r = v_{d+5}$.

Let $H'_r = H_w[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_r = H_w[\{v_{d+3}, v_{d+6}\}]$. Note that $V(H'_r)$ together with $V(H''_r)$ forms a partition of $V(H_r)$.

We claim that no vertex in H'_r is adjacent to a vertex in H''_r . Observe that v_{d+3} is adjacent to neither v_{-2} , v_{-1} , v_1 nor v_2 since $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$, $|(d+3) - (-1)| = d+4 \notin \langle S \rangle$, $|(d+3) - 1| = d+2 \notin \langle S \rangle$ and $|(d+3) - 2| = d+1 \notin \langle S \rangle$. Furthermore, v_{d+6} is adjacent to neither v_{-2} , v_{-1} , v_1 nor v_2 since $|(d+6) - (-2)| = d+8 \notin \langle S \rangle$, $|(d+6) - (-1)| = d+7 \notin \langle S \rangle$, $|(d+6) - 1| = d+5 \notin \langle S \rangle$ and $|(d+6) - 2| = d+4 \notin \langle S \rangle$.

From Case 5.5.16.1, we know that H'_r is well-covered with $\beta(H'_r) = 2$. Furthermore, $v_{d+3} \sim v_{d+6}$ since $|(d+6) - (d+3)| = 3 \in S$, and thus H''_r is well-covered with $\beta(H''_r) = 1$. Hence, H_r is well-covered with $\beta(H_r) = 3$.

Case 5.5.17.4.5 $r = v_{-1}$.

Let $H'_r = H_w[\{v_{d+2}, v_{d+3}, v_{d+5}, v_{d+6}\}]$ and $H''_r = H_w[\{v_{-2}, v_1\}]$. Note that $V(H'_r)$ together with $V(H''_r)$ forms a partition of $V(H_r)$.

We claim that no vertex in H'_r is adjacent to a vertex in H''_r . Observe that v_1 is adjacent to neither v_{d+2} , v_{d+3} , v_{d+5} nor v_{d+6} since $|(d+2) - 1| = d+1 \notin \langle S \rangle$, $|(d+3) - 1| = d+2 \notin \langle S \rangle$, $|(d+5) - 1| = d+4 \notin \langle S \rangle$, and $|(d+6) - 1| = d+5 \notin \langle S \rangle$. Furthermore, v_{-2} is adjacent to neither v_{d+2} , v_{d+3} , v_{d+5} nor v_{d+6} since $|(d+2) - (-2)| = d+4 \notin \langle S \rangle$, $|(d+3) - (-2)| = d+5 \notin \langle S \rangle$, $|(d+5) - (-2)| = d+7 \notin \langle S \rangle$, and $|(d+6) - (-2)| = d+8 \notin \langle S \rangle$.

From Case 5.5.17.4.3, we know that H'_r is well-covered with $\beta(H'_r) = 2$. Furthermore, $v_1 \sim v_{-2}$ since $|1 - (-2)| = 3 \in S$, and thus H''_r is well-covered with $\beta(H''_r) = 1$. Hence, H_r is well-covered with $\beta(H_r) = 3$.

This concludes Case 5.5.17.4.

Case 5.5.17.5 $w = v_k$ or $w = v_{-k}$ for $d + 5 \leq k \leq \lfloor \frac{n}{2} \rfloor - 3$ and $d \geq 10$.

By symmetry, we need only examine $w = v_k$. Since neither 1 nor 2 is in S , we can deduce that v_k is adjacent to neither v_{k-2} , v_{k-1} , v_{k+1} nor v_{k+2} . Let $H'_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$ and $H''_w = H[\{v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. From Case 5.5.16.6, we know that H'_w and H''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = 2$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Case 5.5.17.6 $w = v_{2-\lfloor \frac{n}{2} \rfloor}$. Note that this case is only valid if n is odd.

Let $H'_w = H[\{v_{-\lfloor \frac{n}{2} \rfloor}, v_{1-\lfloor \frac{n}{2} \rfloor}, v_{3-\lfloor \frac{n}{2} \rfloor}, v_{4-\lfloor \frac{n}{2} \rfloor}\}]$ and $H''_w = H[\{v_{-2}, v_{-1}, v_1, v_2\}]$. Note that $V(H'_w)$ together with $V(H''_w)$ forms a partition of $V(H_w)$. From Case 5.5.16.5, we know that H'_w and H''_w are well-covered with $\beta(H'_w) = \beta(H''_w) = 2$. Hence, H_w is well-covered with $\beta(H_w) = 4$.

Hence, G is well-covered with $\beta(G) = 6$, concluding the proof of Case (vii).

(viii) $C(21, \{3\})$.

Note that $C(21, \{3\})$ is well-covered and $\beta(G) = 9$.

We now proceed to prove the ‘only if’ direction.

Case 5.5.18 $d = 3$ and $n \geq 22$.

Observe that for each $n = 10, 11, 13, 14$ and $16 \leq n \leq 20$ one can verify that G is not well-covered.

Assume that $n \geq 22$. Let $I' = \{v_{-9}, v_9\}$. First, we show that $v_{-9} \not\sim v_9$. Note that $|9 - (-9)| = 18$. Given our assumption that $n \geq 22$, it follows that $n - 18 \geq 4$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_3\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_3\}$. Note that $v_{-3} \not\sim v_3$ since $|3 - (-3)| = 6 \notin S$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.19 $d \geq 3$ and $n = 2d + 2$.

First, we note that $C(8, \{3\})$ is not well-covered.

Next, we consider the case where $d \geq 4$. Let $I' = \{v_{d+1}\}$. Clearly I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-(d-1)}, v_{-d}, v_0, v_{d-1}, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_d\}$. Note that $v_{-d} \not\sim v_d$ since $|d - (-d)| = 2d \equiv -2 \pmod{2d+2} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.20 $d \geq 4$ and $3d + 3 \leq n \leq 3d + 5$.

Case 5.5.20.1 $n = 3d + 3$.

Let $I' = \{v_{-(d+1)}, v_{d+1}\}$. Observe that $v_{-(d+1)} \not\sim v_{d+1}$ since $|(d+1) - (-(d+1))| = 2d + 2 \equiv -(d+1) \pmod{3d+3} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-(d-1)}, v_0, v_{d-1}, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_{-(d-1)}, v_{d-1}, v_d\}$. Observe that $v_d \not\sim v_{d-1}$ since $|d - (d-1)| = 1 \notin S$. Next, we note that $v_{-(d-1)}$ is adjacent to neither v_{d-1} nor v_d since $|(d-1) - (-(d-1))| = 2d - 2 \equiv -(d+5) \pmod{3d+3} \notin \langle S \rangle$ and $|d - (-(d-1))| = 2d - 1 \equiv -(d+4) \pmod{3d+3} \notin \langle S \rangle$. Furthermore, $v_d \not\sim v_{-d}$

since $|d - (-d)| = 2d \equiv -(d+3) \pmod{3d+3} \notin \langle S \rangle$. By symmetry, we can also deduce that v_{-d} is adjacent to neither $v_{-(d-1)}$ nor v_{d-1} . Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.20.2 $n = 3d + 4$.

Let $I' = \{v_{-(d+1)}, v_{d+1}, v_{d+2}\}$. Observe that $v_{d+1} \not\sim v_{d+2}$ since $|(d+2) - (d+1)| = 1 \notin S$. Next, we note that $v_{d+2} \not\sim v_{-(d+1)}$ since $|(d+2) - (-(d+1))| = 2d+3 \equiv -(d+1) \pmod{3d+4} \notin \langle S \rangle$. Furthermore, $v_{d+1} \not\sim v_{-(d+1)}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+2) \pmod{3d+4} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_{-(d-1)}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_{-(d-1)}, v_d\}$. Observe that $v_{-(d-1)} \not\sim v_{-d}$ since $|-(d-1) - (-d)| = 1 \notin S$. Next, we note that $v_{-(d-1)} \not\sim v_d$ since $|d - (-(d-1))| = 2d-1 \equiv -(d+5) \pmod{3d+4} \notin \langle S \rangle$. Furthermore, $v_d \not\sim v_{-d}$ since $|d - (-d)| = 2d \equiv -(d+4) \pmod{3d+4} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.20.3 $n = 3d + 5$.

Let $I' = \{v_{-(d+2)}, v_{-(d+1)}, v_{d+1}, v_{d+2}\}$. Observe that $v_{-(d+2)} \not\sim v_{-(d+1)}$ since $|-(d+1) - (-(d+2))| = 1 \notin S$. Next, we note that $v_{-(d+2)}$ is adjacent to neither v_{d+1} nor v_{d+2} since $|(d+1) - (-(d+2))| = 2d+3 \equiv -(d+2) \pmod{3d+5} \notin \langle S \rangle$ and $|(d+2) - (-(d+2))| = 2d+4 \equiv -(d+1) \pmod{3d+5} \notin \langle S \rangle$. Furthermore, $v_{d+1} \not\sim v_{-(d+1)}$ since $|(d+1) - (-(d+1))| = 2d+2 \equiv -(d+3) \pmod{3d+5} \notin \langle S \rangle$. By symmetry, we can also deduce that v_{d+2} is adjacent to neither $v_{-(d+1)}$ nor v_{d+1} . Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-d}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-d}, v_d\}$. Note that $v_{-d} \not\sim v_d$ since $|d - (-d)| = 2d \equiv -(d+5) \pmod{3d+5} \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.21 $d \geq 4$ and $n \geq 3d + 7$.

Case 5.5.21.1 $n = 3d + 7$ and $d \geq 4$.

Let $I' = \{v_{-2}, v_{-1}, v_{2d+1}, v_{2d+3}\}$. Observe that $v_{-2} \not\sim v_{-1}$ since $|-1 - (-2)| = 1 \notin S$, and $v_{2d+1} \not\sim v_{2d+3}$ since $|(2d+3) - (2d+1)| = 2 \notin S$. Next, we note that v_{2d+1} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+1) - (-2)| = 2d+3 \equiv -(d+4) \pmod{3d+7} \notin \langle S \rangle$ and $|(2d+1) - (-1)| = 2d+2 \equiv -(d+5) \pmod{3d+7} \notin \langle S \rangle$. Furthermore, v_{2d+3} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+3) - (-2)| = 2d+5 \equiv -(d+2) \pmod{3d+7} \notin \langle S \rangle$ and $|(2d+3) - (-1)| = 2d+4 \equiv -(d+3) \pmod{3d+7} \notin \langle S \rangle$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_d\}$. Note that $v_{-3} \not\sim v_d$ since $|d - (-3)| = d+3 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.21.2 $d \geq 5$ and $n = 3d + 8$.

Let $I' = \{v_{-2}, v_{-1}, v_{2d+1}, v_{2d+3}\}$. Observe that v_{2d+1} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+1) - (-2)| = 2d+3 \equiv -(d+5) \pmod{3d+8} \notin \langle S \rangle$ and $|(2d+1) - (-1)| = 2d+2 \equiv -(d+6) \pmod{3d+8} \notin \langle S \rangle$. Next, we note that v_{2d+3} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+3) - (-2)| = 2d+5 \equiv -(d+3) \pmod{3d+8} \notin \langle S \rangle$.

$(\text{mod } 3d + 8) \notin \langle S \rangle$ and $|(2d + 3) - (-1)| = 2d + 4 \equiv -(d + 4) \pmod{3d + 8} \notin \langle S \rangle$. Furthermore, from Case 5.5.21.1, we know that $v_{-2} \not\sim v_{-1}$ and $v_{2d+1} \not\sim v_{2d+3}$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_d\}$. Note that $v_{-3} \not\sim v_d$ since $|d - (-3)| = d + 3 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered. A similar argument shows that $n = 3d + 9$ is also not well-covered.

Case 5.5.21.3 $d \geq 6$ and $3d + 10 \leq n \leq 4d + 5$.

Let $I' = \{v_{-2}, v_{-1}, v_{2d+1}, v_{2d+3}\}$. First, we consider v_{2d+1} . Note that $|(2d + 1) - (-2)| = 2d + 3$ and $|(2d + 1) - (-1)| = 2d + 2$. Given our assumption that $3d + 10 \leq n \leq 4d + 6$, it follows that $d + 7 \leq n - (2d + 3) \leq 2d + 3$ and $d + 8 \leq n - (2d + 2) \leq 2d + 4$. Hence, v_{2d+1} is adjacent to neither v_{-2} nor v_{-1} . Next, we consider v_{2d+3} . Note that $|(2d + 3) - (-2)| = 2d + 5$ and $|(2d + 3) - (-1)| = 2d + 4$. Given our assumption that $3d + 10 \leq n \leq 4d + 6$, it follows that $d + 5 \leq n - (2d + 5) \leq 2d + 1$ and $d + 6 \leq n - (2d + 4) \leq 2d + 2$. Hence, v_{2d+3} is adjacent to neither v_{-2} nor v_{-1} . Furthermore, from Case 5.5.21.1, we know that $v_{-2} \not\sim v_{-1}$ and $v_{2d+1} \not\sim v_{2d+3}$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_d\}$. Note that $v_{-3} \not\sim v_d$ since $|d - (-3)| = d + 3 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.21.4 $d \geq 7$ and $n = 4d + 6$.

Let $I' = \{v_{-2}, v_{-1}, v_{2d+1}, v_{2d+3}\}$. Observe that v_{2d+1} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+1) - (-2)| = 2d+3 \notin \langle S \rangle$ and $|(2d+1) - (-1)| = 2d+2 \notin \langle S \rangle$. Next, we note that v_{2d+3} is adjacent to neither v_{-2} nor v_{-1} since $|(2d+3) - (-2)| = 2d+5 \notin \langle S \rangle$ and $|(2d+3) - (-1)| = 2d+4 \notin \langle S \rangle$. Furthermore, from Case 5.5.21.1, we know that $v_{-2} \not\sim v_{-1}$ and $v_{2d+1} \not\sim v_{2d+3}$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_d\}$. Note that $v_{-3} \not\sim v_d$ since $|d - (-3)| = d+3 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.21.5 $d \geq 8$ and $4d + 7 \leq n \leq 5d - 1$.

Let $I' = \{v_{-(d+3)}, v_{-(d+2)}, v_{-(d+1)}, v_{d+1}, v_{2d+2}\}$. Observe that $v_{-(d+3)}$ is adjacent to neither $v_{-(d+2)}$, $v_{-(d+1)}$ nor v_{d+1} since $|-(d+2) - (-(d+3))| = 1 \notin S$, $|-(d+1) - (-(d+3))| = 2 \notin S$ and $|(d+1) - (-(d+3))| = 2d+4 \notin \langle S \rangle$. Next, we note that $v_{-(d+1)}$ is adjacent to neither $v_{-(d+2)}$ nor v_{d+1} since $|-(d+1) - (-(d+2))| = 1 \notin S$ and $|(d+1) - (-(d+1))| = 2d+2 \notin \langle S \rangle$. We also note that v_{d+1} is adjacent to neither $v_{-(d+2)}$ nor v_{2d+2} since $|(d+1) - (-(d+2))| = 2d+3 \notin \langle S \rangle$ and $|(2d+2) - (d+1)| = d+1 \notin \langle S \rangle$. Finally, we consider v_{2d+2} . Note that $|(2d+2) - (-(d+3))| = 3d+5$, $|(2d+2) - (-(d+2))| = 3d+4$ and $|(2d+2) - (-(d+1))| = 3d+3$. Given our assumption that $4d+7 \leq n \leq 5d-1$, it follows that $d+2 \leq n - (3d+5) \leq 2d-6$, $d+3 \leq n - (3d+4) \leq 2d-5$, and $d+4 \leq n - (3d+3) \leq 2d-4$. Hence, v_{2d+2} is adjacent to neither $v_{-(d+3)}$, $v_{-(d+2)}$ nor $v_{-(d+1)}$. Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_0, v_{d-1}, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{d-1}, v_d\}$. Note that $v_{d-1} \not\sim v_d$ since $|d - (d-1)| = 1 \notin S$. Therefore,

K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

Case 5.5.21.6 $d \geq 4$ and $n \geq 5d$.

Let $I' = \{v_{-(d+6)}, v_{-2}, v_{-1}, v_{2d+1}, v_{2d+3}\}$. Observe that v_{-1} is adjacent to neither $v_{-(d+6)}$, v_{-2} , v_{2d+1} nor v_{2d+3} since $|(-1) - (-(d+6))| = d+5 \notin \langle S \rangle$, $|(-1) - (-2)| = 1 \notin S$, $|(2d+1) - (-1)| = 2d+2 \notin \langle S \rangle$ and $|(2d+3) - (-1)| = 2d+4 \notin \langle S \rangle$. Next, we note that v_{-2} is adjacent to neither $v_{-(d+6)}$, v_{2d+1} nor v_{2d+3} since $|(-2) - (-(d+6))| = d+4 \notin \langle S \rangle$, $|(2d+1) - (-2)| = 2d+3 \notin \langle S \rangle$ and $|(2d+3) - (-2)| = 2d+5 \notin \langle S \rangle$. Also note that $v_{2d+1} \not\sim v_{2d+3}$ since $|(2d+3) - (2d+1)| = 2 \notin S$. Finally, we consider $v_{-(d+6)}$. Note that $|(2d+1) - (-(d+6))| = 3d+7$ and $|(2d+3) - (-(d+6))| = 3d+9$. Given our assumption that $n \geq 5d$, it follows that $n - (3d+7) \geq 2d-7$ and $n - (3d+9) \geq 2d-9$. Hence, $v_{-(d+6)}$ is adjacent to neither v_{2d+1} nor v_{2d+3} . Hence, I' is an independent set in G .

Now let H_1 be the component of $G \setminus N[I']$ containing v_0 . It follows that $V(H_1) = \{v_{-3}, v_0, v_d\}$. First, let $K_1 = \{v_0\}$. Clearly K_1 is a maximal independent set in H_1 . Next, let $K_2 = \{v_{-3}, v_d\}$. Note that $v_{-3} \not\sim v_d$ since $|d - (-3)| = d+3 \notin \langle S \rangle$. Therefore, K_2 is an independent set in H_1 with cardinality greater than that of K_1 . So H_1 is not well-covered, and hence by Proposition 2.5, G is not well-covered.

■

Chapter 6

Characterization of Well-Covered Graphs in Classes 11, 12, and 13

In this chapter, we investigate the class of circulant graphs on n vertices with a generating set $S - A$, where A is a subset of S such that A is of size one, two, or three and $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. To determine the well-coveredness of G it turns out to be useful to work with the maximal cliques in \overline{G} . Note that the set of vertices of a maximal independent set of G is a maximal clique in \overline{G} . The converse is true as well. Hence, one could choose either point of view to tackle the property of well-coveredness.

This is done via the following:

Proposition 6.1 *A graph G is well-covered if and only if every maximal clique in the complement graph, \overline{G} , has the same size.*

Proof. Suppose that every maximal clique in \overline{G} is of the same size. Since every maximal clique in \overline{G} corresponds to a maximal independent set in G , it follows that all maximal independent sets are of the same size in G , thus G is well-covered.

Next, assume that G is well-covered. By definition, every maximal independent set is of the same size. Since every independent set in G corresponds to a maximal clique in \overline{G} , it follows that all maximal cliques are of the same size in \overline{G} . ■

Proposition 6.2 *Let $G = C(n, S)$ be a circulant graph on n vertices. Then $\overline{G} = C(n, \overline{S})$ is also circulant.*

Corollary 6.3 *Let $G = C(n, S)$ be a circulant graph on n vertices and let v be an arbitrary vertex in \overline{G} . If all maximal cliques in \overline{G} containing v are of the same size then G is well-covered.*

Now we can determine which graphs in Class 11 (see Section 1.2) are well-covered.

Theorem 6.4 *Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$, where $A \subseteq S$ and $|A| = 1$. Then G is well-covered. Furthermore, $\beta(G) = 2$ unless $\{\frac{n}{3}\} \in A$ in which case $\beta(G) = 3$.*

Proof. Let $V(G) = \{v_i: i = 0, 1, \dots, n-1\}$. By Corollary 6.2, $\overline{G} = C(n, \overline{S})$. Let $A = \{x\}$ and without loss of generality assume that $0 < x \leq \frac{n}{2}$. Note that $\overline{S} = A$. By Corollary 6.3, it suffices to show that all maximal cliques of \overline{G} containing v_0 are of the same size. Since v_0 is adjacent to v_{0+j} and v_{0-j} for each j in the set \overline{S} , $N_{\overline{G}}[v_0] = \{v_{-x}, v_x\}$.

Case 6.4.1 $v_x \sim v_{-x}$.

Note that $v_x \sim v_{-x}$ in \overline{G} if and only if $|2x| \pmod{n} \in \langle \overline{S} \rangle$; that is, $2x \equiv x \pmod{n}$ or $2x \equiv -x \pmod{n}$. Observe that $x = 0$ contradicts our assumption that $x > 0$, hence $3x = n$. Thus, we have exactly one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.4.2 $v_x = v_{-x}$.

Note that $v_x = v_{-x}$ in \overline{G} if and only if $|2x| \equiv 0 \pmod{n}$, and thus $2x = n$. Therefore, we have exactly one 2-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{2}\}$, G is well-covered and $\beta(G) = 2$.

Case 6.4.3 $v_x \not\sim v_{-x}$ and $\{\frac{n}{2}, \frac{n}{3}\} \cap A = \emptyset$.

Note that $v_x \not\sim v_{-x}$ in \overline{G} if and only if $|2x| \pmod{n} \notin \langle \overline{S} \rangle$. Thus, we have exactly two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, G is well-covered and $\beta(G) = 2$.

■

Next, employing a similar approach we characterize our second family in which A is of cardinality two. However, the following two lemmas are needed prior to stating our results.

Lemma 6.5 *Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{x, y\}$, where without loss of generality $0 < x < y \leq \frac{n}{2}$. Then*

- (i) $v_x \sim v_y$ and $v_{-x} \sim v_{-y}$ if and only if $y = 2x$;
- (ii) $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$ if and only if $2y + x = n$ or $2x + y = n$;
- (iii) $v_x \sim v_{-x}$ if and only if $y = 2x$, $2x + y = n$ or $3x = n$;
- (iv) $v_y \sim v_{-y}$ if and only if $2y + x = n$ or $3y = n$;
- (v) $v_y = v_{-y}$ if and only if $2y = n$.

Proof.

- (i) $v_x \sim v_y$ in G if and only if $|y - x| \pmod{n} \in \langle S \rangle$.

First note that $v_{-x} \sim v_{-y}$ if and only if $v_x \sim v_y$. Now $v_x \sim v_y$ implies at least one of the following:

- (a) $y - x \equiv -x \pmod{n}$ but $y \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq y > 0$.
- (b) $y - x \equiv y \pmod{n}$ but $x \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} > x > 0$.
- (c) $y - x \equiv -y \pmod{n}$ but $x \equiv 2y \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq y > x$.

$$(d) \ y - x \equiv x \pmod{n}.$$

Hence, $y = 2x$ is the only possibility, completing case (i).

$$(ii) \ v_x \sim v_{-y} \text{ in } G \text{ if and only if } |x + y| \pmod{n} \in \langle S \rangle.$$

First note that $v_{-x} \sim v_y$ if and only if $v_x \sim v_{-y}$. Now $v_x \sim v_{-y}$ implies at least one of the following:

$$(a) \ x + y \equiv x \pmod{n} \text{ but } y \equiv 0 \pmod{n} \text{ contradicts the assumption that}$$

$$\frac{n}{2} \geq y > 0.$$

$$(b) \ x + y \equiv y \pmod{n} \text{ but } x \equiv 0 \pmod{n} \text{ contradicts the assumption that}$$

$$\frac{n}{2} > x > 0.$$

$$(c) \ x + y \equiv -x \pmod{n}.$$

$$(d) \ x + y \equiv -y \pmod{n}.$$

Hence, $2x + y = n$ and $2y + x = n$ are the only possibilities, completing case (ii).

$$(iii) \ v_x \sim v_{-x} \text{ in } G \text{ if and only if } |2x| \pmod{n} \in \langle S \rangle.$$

Note that $v_x \sim v_{-x}$ implies at least one of the following:

$$(a) \ 2x \equiv x \pmod{n} \text{ but } x \equiv 0 \pmod{n} \text{ contradicts the assumption that } \frac{n}{2} >$$

$$x > 0.$$

$$(b) \ 2x \equiv -x \pmod{n}.$$

$$(c) \ 2x \equiv y \pmod{n}.$$

$$(d) \ 2x \equiv -y \pmod{n}.$$

Hence, $3x = n$, $y = 2x$ and $2x + y = n$ are the only possibilities, completing case (iii).

(iv) $v_y \sim v_{-y}$ in G if and only if $|2y| \pmod n \in \langle S \rangle$.

Note that $v_y \sim v_{-y}$ implies at least one of the following:

(a) $2y \equiv x \pmod n$ but $x \equiv 2y \pmod n$ contradicts the assumption that

$$\frac{n}{2} \geq y > x.$$

(b) $2y \equiv y \pmod n$ but $y \equiv 0 \pmod n$ contradicts the assumption that $\frac{n}{2} \geq$

$$y > 0.$$

(c) $2y \equiv -x \pmod n$.

(d) $2y \equiv -y \pmod n$.

Hence, $2y + x = n$ and $3y = n$ are the only possibilities, completing case (iv).

(v) $v_y = v_{-y}$ in G if and only if $|2y| \equiv 0 \pmod n$, and thus $2y = n$, completing case (v).

■

Lemma 6.6 *Suppose x, y and $z \in \mathbb{Z}_n$. In each of the following, the given set of equations is inconsistent with the inequalities $0 < x < y \leq \frac{n}{2}$.*

(i) $3x = n$ and $y = 2x$;

(ii) $3x = n$ and $2x + y = n$;

(iii) $3x = n$ and $2y + x = n$;

(iv) $3x = n$ and $3y = n$;

(v) $3y = n$ and $2x + y = n$;

(vi) $3y = n$ and $2y + x = n$;

- (vii) $3y = n$ and $2y = n$;
- (viii) $2y = n$ and $2y + x = n$;
- (ix) $2y + x = n$ and $2x + y = n$.

Proof. Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{x, y\}$. Without loss of generality assume that $0 < x < y \leq \frac{n}{2}$.

- (i) If $3x = n$ and $y = 2x$, then $x = \frac{n}{3}$ and $y = \frac{2n}{3}$, a contradiction since $y \leq \frac{n}{2}$.
- (ii) If $3x = n$ and $2x + y = n$, then $x = y$, a contradiction since $x < y$.
- (iii) If $3x = n$ and $2y + x = n$, then $x = y$, a contradiction since $x < y$.
- (iv) If $3x = n$ and $3y = n$, then $x = y$, a contradiction since $x < y$.
- (v) If $3y = n$ and $2x + y = n$, then $x = y$, a contradiction since $x < y$.
- (vi) If $3y = n$ and $2y + x = n$, then $x = y$, a contradiction since $x < y$.
- (vii) If $3y = n$ and $2y = n$, then $y = 0$, a contradiction since $y > 0$.
- (viii) If $2y = n$ and $2y + x = n$, then $x = 0$, a contradiction since $x > 0$.
- (ix) If $2y + x = n$ and $2x + y = n$, then $x = y$, a contradiction since $x < y$.

■

Now we can determine which graphs in Class 12 (see Section 1.2) are well-covered.

Theorem 6.7 *Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$, where $A \subseteq S$ and $|A| = 2$. Let $A = \{x, y\}$ where $0 < x < y \leq \frac{n}{2}$.*

- (i) If n is not divisible by three, then G is well-covered.
- (ii) If n is divisible by three, then G is well-covered if and only if one of the following is true:

(a) $\frac{n}{3} \notin A$, or

(b) $A = \{\frac{n}{6}, \frac{n}{3}\}$.

Furthermore,

- $\beta(G) = 2$ if either

$$y = \frac{n}{2}; \text{ or}$$

if all of $y \neq 2x$, $2y + x \neq n$, $2x + y \neq n$ and $2y \neq n$ hold;

- $\beta(G) = 3$ if A is any one of

$$\{n - 2y, y\},$$

$$\{x, n - 2x\},$$

$$\{x, 2x\}, \text{ or}$$

$$\{\frac{n}{6}, \frac{n}{3}\};$$

- $\beta(G) = 4$ if $A = \{\frac{n}{4}, \frac{n}{2}\}$; and

- $\beta(G) = 5$ if $A = \{\frac{n}{5}, \frac{2n}{5}\}$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. By Corollary 6.2, $\overline{G} = C(n, \overline{S})$. Let $A = \{x, y\}$ and without loss of generality assume that $0 < x < y \leq \frac{n}{2}$. Note that $\overline{S} = A$. By Corollary 6.3, it suffices to show that all maximal cliques of \overline{G} containing v_0 are of the same size. Since v_0 is adjacent to v_{0+j} and v_{0-j} for each j in the set \overline{S} , $N_{\overline{G}}[v_0] = \{v_{-y}, v_{-x}, v_x, v_y\}$.

Case 6.7.1 $3x = n$.

By Lemma 6.5, $v_x \sim v_{-x}$, and by Lemma 6.6, $y \neq 2x$, $2y + x \neq n$, $2x + y \neq n$ and $3y \neq n$. We consider two cases.

Case 6.7.1.1 $2y = n$.

By Lemma 6.5, $v_y = v_{-y}$. Thus, we have one 3-vertex maximal clique and one 2-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{3}, \frac{n}{2}\}$ and G is not well-covered.

Case 6.7.1.2 $2y \neq n$.

By Lemma 6.5, $v_y \neq v_{-y}$. Thus, we have one 3-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{3}, y\}$ and G is not well-covered.

Case 6.7.2 $3x \neq n$.

We consider two cases.

Case 6.7.2.1 $y \neq 2x$.

We consider two cases.

Case 6.7.2.1.1 $2y + x = n$.

By Lemma 6.5, $v_x \sim v_{-y}$, $v_y \sim v_{-x}$ and $v_y \sim v_{-y}$, and by Lemma 6.6, $2x + y \neq n$, $2y \neq n$ and $3y \neq n$. Thus, we have exactly three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{n - 2y, y\}$, G is well-covered and $\beta(G) = 3$.

Case 6.7.2.1.2 $2y + x \neq n$.

We consider two cases.

Case 6.7.2.1.2.1 $2x + y = n$.

By Lemma 6.5, $v_x \sim v_{-y}$, $v_y \sim v_{-x}$ and $v_x \sim v_{-x}$, and by Lemma 6.6, $3y \neq n$. We also note that $2x + y = n$ together with $2y = n$ is inconsistent with our assumption that $y \neq 2x$, and hence $2y \neq n$ and by Lemma 6.5 $v_y \neq v_{-y}$. Thus, we have exactly three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, n - 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.7.2.1.2.2 $2x + y \neq n$.

We consider two cases.

Case 6.7.2.1.2.2.1 $3y = n$.

By Lemma 6.5, $v_y \sim v_{-y}$, and by Lemma 6.6, $2y \neq n$. Thus, we have one 3-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}\}$ and G is not well-covered.

Case 6.7.2.1.2.2.2 $3y \neq n$.

We consider two cases.

Case 6.7.2.1.2.2.2.1 $2y = n$.

By Lemma 6.5, $v_y = v_{-y}$. Thus, we have exactly three 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 2$.

Case 6.7.2.1.2.2.2.2 $2y \neq n$.

By Lemma 6.5, $v_y \neq v_{-y}$. Thus, we have exactly four 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, G is well-covered and $\beta(G) = 2$.

Case 6.7.2.2 $y = 2x$.

By Lemma 6.5, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. We consider two cases.

Case 6.7.2.2.1 $2y + x = n$.

By Lemma 6.5, $v_x \sim v_{-y}$, $v_y \sim v_{-x}$ and $v_y \sim v_{-y}$, and by Lemma 6.6, $2x + y \neq n$, $2y \neq n$ and $3y \neq n$. Thus, we have exactly one 5-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{5}, \frac{2n}{5}\}$, G is well-covered and $\beta(G) = 5$.

Case 6.7.2.2.2 $2y + x \neq n$.

We consider two cases.

Case 6.7.2.2.2.1 $2x + y = n$.

By Lemma 6.5, $v_x \sim v_{-y}$, $v_y \sim v_{-x}$ and $v_x \sim v_{-x}$, and by Lemma 6.6, $3y \neq n$. Given our assumption that $y = 2x$, we can also deduce that $2y = n$, and hence, by Lemma 6.5, $v_y = v_{-y}$. Thus, we have exactly one 4-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{4}, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 4$.

Case 6.7.2.2.2.2 $2x + y \neq n$.

We consider two cases.

Case 6.7.2.2.2.2.1 $3y = n$.

By Lemma 6.5, $v_y \sim v_{-y}$, and by Lemma 6.6, $2y \neq n$. Thus, we have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{6}, \frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.7.2.2.2.2.2 $3y \neq n$.

Note that $y = 2x$ together with $2y = n$ is inconsistent with our assumption that $2x + y \neq n$, and hence $2y \neq n$. Thus, we have exactly three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x\}$, G is well-covered and $\beta(G) = 3$.

■

We turn our focus to the case where A is of cardinality three. The following two lemmas are needed prior to stating our main results.

Lemma 6.8 *Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{x, y, z\}$, where without loss of generality $0 < x < y < z \leq \frac{n}{2}$. Then*

- (i) $v_x \sim v_y$ and $v_{-x} \sim v_{-y}$ if and only if $y = 2x$;
- (ii) $v_y \sim v_z$ and $v_{-y} \sim v_{-z}$ if and only if $z = 2y$ or $z = x + y$;
- (iii) $v_x \sim v_z$ and $v_{-x} \sim v_{-z}$ if and only if $z = 2x$ or $z = x + y$;
- (iv) $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$ if and only if $z = x + y$, $2x + y = n$, $2y + x = n$ or $x + y + z = n$;
- (v) $v_x \sim v_{-z}$ and $v_{-x} \sim v_z$ if and only if $2z + x = n$, $2x + z = n$ or $x + y + z = n$;
- (vi) $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$ if and only if $2y + z = n$, $2z + y = n$ or $x + y + z = n$;
- (vii) $v_x \sim v_{-x}$ if and only if $2x + y = n$, $2x + z = n$, $y = 2x$, $z = 2x$ or $3x = n$;
- (viii) $v_y \sim v_{-y}$ if and only if $2y + x = n$, $2y + z = n$, $z = 2y$ or $3y = n$;
- (ix) $v_z \sim v_{-z}$ if and only if $2z + x = n$ or $2z + y = n$ or $3z = n$;
- (x) $v_z = v_{-z}$ if and only if $2z = n$.

Proof.

- (i) $v_x \sim v_y$ in G if and only if $|y - x| \pmod{n} \in \langle S \rangle$.

First note that $v_{-x} \sim v_{-y}$ if and only if $v_x \sim v_y$. Now $v_x \sim v_y$ implies at least one of the cases looked at in the proof of Lemma 6.5 (Case (i)) together with the following.

- (a) $y - x \equiv z \pmod{n}$ but $y \equiv x + z \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > y.$$

- (b) $y - x \equiv -z \pmod{n}$ but $x \equiv y + z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > y > x$.

Hence, by this and the proof of Lemma 6.5, $y = 2x$ is the only possibility, completing case (i).

- (ii) $v_y \sim v_z$ in G if and only if $|z - y| \pmod{n} \in \langle S \rangle$.

First note that $v_{-y} \sim v_{-z}$ if and only if $v_y \sim v_z$. Now $v_y \sim v_z$ implies at least one of the following:

- (a) $z - y \equiv -x \pmod{n}$ but $y \equiv x + z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > y$.
- (b) $z - y \equiv -y \pmod{n}$ but $z \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > 0$.
- (c) $z - y \equiv -z \pmod{n}$ but $y \equiv 2z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > y$.
- (d) $z - y \equiv z \pmod{n}$ but $y \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} > y > 0$.
- (e) $z - y \equiv x \pmod{n}$.
- (f) $z - y \equiv y \pmod{n}$.

Hence, $z = y + x$ and $z = 2y$ are the only possibilities, completing case (ii).

- (iii) $v_x \sim v_z$ in G if and only if $|z - x| \pmod{n} \in \langle S \rangle$.

First note that $v_{-x} \sim v_{-z}$ if and only if $v_x \sim v_z$. Now $v_x \sim v_z$ implies at least one of the following:

- (a) $z - x \equiv -x \pmod{n}$ but $z \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > 0$.

(b) $z - x \equiv -y \pmod{n}$ but $x \equiv y + z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > x$.

(c) $z - x \equiv z \pmod{n}$ but $x \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} > x > 0$.

(d) $z - x \equiv -z \pmod{n}$ but $x \equiv 2z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq z > x$.

(e) $z - x \equiv x \pmod{n}$.

(f) $z - x \equiv y \pmod{n}$.

Hence, $z = 2x$ and $z = x + y$ are the only possibilities, completing case (iii).

(iv) $v_x \sim v_{-y}$ in G if and only if $|x + y| \pmod{n} \in \langle S \rangle$.

First note that $v_{-x} \sim v_y$ if and only if $v_x \sim v_{-y}$. Now $v_x \sim v_{-y}$ implies at least one of the cases looked at in the proof of Lemma 6.5 (Case (ii)) together with the following.

(a) $x + y \equiv z \pmod{n}$.

(b) $x + y \equiv -z \pmod{n}$.

Hence, by this and by the proof of Lemma 6.5, $z = x + y$, $x + y + z = n$, $2x + y = n$ and $2y + x = n$ are the only possibilities, completing case (iv).

(v) $v_x \sim v_{-z}$ in G if and only if $|x + z| \pmod{n} \in \langle S \rangle$.

First note that $v_{-x} \sim v_z$ if and only if $v_x \sim v_{-z}$. Now $v_x \sim v_{-z}$ implies at least one of the following:

(a) $x + z \equiv x \pmod{n}$ but $z \equiv 0 \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > 0.$$

(b) $x + z \equiv y \pmod{n}$ but $y \equiv x + z \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > y.$$

(c) $x + z \equiv z \pmod{n}$ but $x \equiv 0 \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} > x > 0.$$

(d) $x + z \equiv -x \pmod{n}$.

(e) $x + z \equiv -y \pmod{n}$.

(f) $x + z \equiv -z \pmod{n}$.

Hence, $2x + z = n$, $x + y + z = n$ and $2z + x = n$ are the only possibilities, completing case (v).

(vi) $v_y \sim v_{-z}$ in G if and only if $|y + z| \pmod{n} \in \langle S \rangle$.

First note that $v_{-y} \sim v_z$ if and only if $v_y \sim v_{-z}$. Now $v_y \sim v_{-z}$ implies at least one of the following:

(a) $y + z \equiv x \pmod{n}$ but $x \equiv y + z \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > x.$$

(b) $y + z \equiv y \pmod{n}$ but $z \equiv 0 \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > 0.$$

(c) $y + z \equiv z \pmod{n}$ but $y \equiv 0 \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} > y > 0.$$

(d) $y + z \equiv -x \pmod{n}$.

(e) $y + z \equiv -y \pmod{n}$.

(f) $y + z \equiv -z \pmod{n}$.

Hence, $x + y + z = n$, $2y + z = n$ and $2z + y = n$ are the only possibilities, completing case (vi).

(vii) $v_x \sim v_{-x}$ in G if and only if $|2x| \pmod{n} \in \langle S \rangle$.

Note that $v_x \sim v_{-x}$ implies at least one of the cases looked at in the proof of Lemma 6.5 (Case (iii)) together with the following.

(a) $2x \equiv z \pmod{n}$.

(b) $2x \equiv -z \pmod{n}$.

Hence, by this and by the proof of Lemma 6.5, $z = 2x$, $2x + z = n$, $y = 2x$, $2x + y = n$ and $3x = n$ are the only possibilities, completing case (vii).

(viii) $v_y \sim v_{-y}$ in G if and only if $|2y| \pmod{n} \in \langle S \rangle$.

Note that $v_y \sim v_{-y}$ implies at least one of the cases looked at in the proof of Lemma 6.5 (Case (iv)) together with the following.

(a) $2y \equiv z \pmod{n}$.

(b) $2y \equiv -z \pmod{n}$.

Hence, by this and the proof of Lemma 6.5, $z = 2y$, $2y + z = n$, $3y = n$ and $2y + x = n$ are the only possibilities, completing case (viii).

(ix) $v_z \sim v_{-z}$ in G if and only if $|2z| \pmod{n} \in \langle S \rangle$.

Note that $v_z \sim v_{-z}$ implies at least one of the following:

(a) $2z \equiv x \pmod{n}$ but $x \equiv 2z \pmod{n}$ contradicts the assumption that

$$\frac{n}{2} \geq z > x.$$

(b) $2z \equiv y \pmod{n}$ but $y \equiv 2z \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq$

$$z > y.$$

(c) $2z \equiv z \pmod{n}$ but $z \equiv 0 \pmod{n}$ contradicts the assumption that $\frac{n}{2} \geq$

$$z > 0.$$

$$(d) \ 2z \equiv -x \pmod{n}.$$

$$(e) \ 2z \equiv -y \pmod{n}.$$

$$(f) \ 2z \equiv -z \pmod{n}.$$

Hence, $2z + x = n$, $2z + y = n$ and $3z = n$ are the only possibilities, completing case (ix).

(x) $v_z = v_{-z}$ in G if and only if $|2z| \equiv 0 \pmod{n}$, and thus $2z = n$, completing case (x).

■

Lemma 6.9 *Suppose x, y and $z \in \mathbb{Z}_n$. In each of the following, the given set of equations is inconsistent with the inequalities $0 < x < y < z \leq \frac{n}{2}$.*

$$(i) \ y = 2x \text{ and } z = 2x;$$

$$(ii) \ y = 2x \text{ and } 2x + z = n;$$

$$(iii) \ 2z = n \text{ and } 2z + x = n;$$

$$(iv) \ 2z = n \text{ and } 2z + y = n;$$

$$(v) \ 2z = n \text{ and } 3z = n;$$

$$(vi) \ z = x + y \text{ and } z = 2y;$$

$$(vii) \ z = x + y \text{ and } z = 2x;$$

$$(viii) \ z = x + y \text{ and } 2y + x = n;$$

$$(ix) \ z = 2x \text{ and } z = 2y;$$

$$(x) \ z = 2y \text{ and } 2y + x = n;$$

- (xi) $3x = n$ and $z = x + y$;
- (xii) $3y = n$ and $z = 2y$;
- (xiii) $2z + x = n$ and $x + y + z = n$;
- (xiv) $2z + x = n$ and $2x + z = n$;
- (xv) $2z + x = n$ and $2y + x = n$;
- (xvi) $2z + x = n$ and $2z + y = n$;
- (xvii) $x + y + z = n$ and $2x + z = n$;
- (xviii) $x + y + z = n$ and $2y + x = n$;
- (xix) $x + y + z = n$ and $2z + y = n$;
- (xx) $x + y + z = n$ and $2y + z = n$;
- (xxi) $2y + z = n$ and $2x + z = n$;
- (xxii) $2y + z = n$ and $2y + x = n$;
- (xxiii) $2y + z = n$ and $2z + y = n$;
- (xxiv) $3x = n$ and $2x + z = n$;
- (xxv) $3z = n$ and $2x + z = n$;
- (xxvi) $3x = n$ and $2z + x = n$;
- (xxvii) $3z = n$ and $2z + x = n$;
- (xxviii) $3y = n$ and $3x = n$;
- (xxix) $3z = n$ and $3x = n$;

- (xxx) $3y = n$ and $2z + y = n$;
- (xxxi) $3z = n$ and $2z + y = n$;
- (xxxii) $3z = n$ and $3y = n$;
- (xxxiii) $3y = n$ and $2y + z = n$;
- (xxxiv) $3z = n$ and $2y + z = n$;
- (xxv) $y = 2x$ and $2x + y = n$;
- (xxxvi) $2y + z = n$ and $3x = n$;
- (xxxvii) $2z + y = n$ and $2x + y = n$;
- (xxxviii) $z = x + y$ and $2x + y = n$;
- (xxxix) $2y + x = n$ and $2x + y = n$;
- (xl) $z = 2y$ and $3x = n$;
- (xli) $z = 2y$ and $2x + z = n$;
- (xlii) $z = 2y$ and $2x + y = n$;
- (xliii) $z = 2y$ and $x + y + z = n$;
- (xliv) $z = 2x$ and $2x + y = n$;
- (xlv) $3x = n$ and $z = 2x$;
- (xlvi) $3z = n$ and $2y + x = n$;
- (xlvii) $3z = n$ and $2x + y = n$;
- (xlviii) $3x = n$ and $x + y + z = n$;

- (xlx) $3z = n$ and $x + y + z = n$;
- (l) $2x + z = n$ and $2x + y = n$;
- (li) $2x + y = n$ and $x + y + z = n$;
- (lii) $3x = n$ and $2z + y = n$;
- (liii) $2x + y = n$ and $2y + z = n$;
- (liv) $2x + z = n$ and $2z + y = n$;
- (lv) $2y + x = n$ and $2z + y = n$;
- (lvi) $2z + x = n$ and $2x + y = n$;
- (lvii) $z = x + y$ and $2x + z = n$;
- (lviii) $2y + z = n$, $z = 2x$ and $2z = n$;
- (lix) $2y + z = n$, $z = x + y$ and $2z = n$;
- (lx) $x + y + z = n$, $z = 2x$ and $2z = n$;
- (lxi) $z = 2y$, $2y + z = n$ and $2z + x = n$;
- (lxii) $2z + x = n$, $z = x + y$, $y = 2x$ and $3y = n$.

Proof. Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{x, y, z\}$. Without loss of generality assume that $0 < x < y < z \leq \frac{n}{2}$.

- (i) If $y = 2x$ and $z = 2x$, then $y = z$, a contradiction since $y < z$.
- (ii) If $y = 2x$ and $2x + z = n$, then $z = n - y$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $n - y \leq \frac{n}{2}$, and thus $y \geq \frac{n}{2}$. This is a contradiction since $y < z \leq \frac{n}{2}$.

- (iii) If $2z = n$ and $2z + x = n$, then $x = 0$, a contradiction since $x > 0$.
- (iv) If $2z = n$ and $2z + y = n$, then $y = 0$, a contradiction since $y > 0$.
- (v) If $2z = n$ and $3z = n$, then $z = 0$, a contradiction since $z > 0$.
- (vi) If $z = x + y$ and $z = 2y$, then $x = y$, a contradiction since $x < y$.
- (vii) If $z = x + y$ and $z = 2x$, then $x = y$, a contradiction since $x < y$.
- (viii) If $z = x + y$ and $2y + x = n$, then $z = \frac{n}{2} + \frac{x}{2}$, a contradiction since $z \leq \frac{n}{2}$.
- (ix) If $z = 2x$ and $z = 2y$, then $x = y$, a contradiction since $x < y$.
- (x) If $z = 2y$ and $2y + x = n$, then $z = n - x$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $n - x \leq \frac{n}{2}$, and thus $x \geq \frac{n}{2}$. This is a contradiction since $x < z \leq \frac{n}{2}$.
- (xi) If $3x = n$ and $z = x + y$, then $x = \frac{n}{3}$ and $z = \frac{n}{3} + y$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $\frac{n}{3} + y \leq \frac{n}{2}$, and thus $y \leq \frac{n}{6}$. This is a contradiction since $x < y$.
- (xii) If $3y = n$ and $z = 2y$, then $y = \frac{n}{3}$ and $z = \frac{2n}{3}$, a contradiction since $z \leq \frac{n}{2}$.
- (xiii) If $2z + x = n$ and $x + y + z = n$, then $y = z$, a contradiction since $y < z$.
- (xiv) If $2z + x = n$ and $2x + z = n$, then $x = z$, a contradiction since $x < z$.
- (xv) If $2z + x = n$ and $2y + x = n$, then $y = z$, a contradiction since $y < z$.
- (xvi) If $2z + x = n$ and $2z + y = n$, then $x = y$, a contradiction since $x < y$.
- (xvii) If $x + y + z = n$ and $2x + z = n$, then $x = y$, a contradiction since $x < y$.
- (xviii) If $x + y + z = n$ and $2y + x = n$, then $y = z$, a contradiction since $y < z$.
- (xix) If $x + y + z = n$ and $2z + y = n$, then $x = z$, a contradiction since $x < z$.

- (xx) If $x + y + z = n$ and $2y + z = n$, then $x = y$, a contradiction since $x < y$.
- (xxi) If $2y + z = n$ and $2x + z = n$, then $x = y$, a contradiction since $x < y$.
- (xxii) If $2y + z = n$ and $2y + x = n$, then $x = z$, a contradiction since $x < z$.
- (xxiii) If $2y + z = n$ and $2z + y = n$, then $y = z$, a contradiction since $y < z$.
- (xxiv) If $3x = n$ and $2x + z = n$, then $x = \frac{n}{3}$ and $z = \frac{n}{3}$, a contradiction since $x < z$.
- (xxv) If $3z = n$ and $2x + z = n$, then $z = \frac{n}{3}$ and $x = \frac{n}{3}$, a contradiction since $x < z$.
- (xxvi) If $3x = n$ and $2z + x = n$, then $x = \frac{n}{3}$ and $z = \frac{n}{3}$, a contradiction since $x < z$.
- (xxvii) If $3z = n$ and $2z + x = n$, then $z = \frac{n}{3}$ and $x = \frac{n}{3}$, a contradiction since $x < z$.
- (xxviii) If $3y = n$ and $3x = n$, then $x = y$, a contradiction since $x < y$.
- (xxix) If $3z = n$ and $3x = n$, then $x = z$, a contradiction since $x < z$.
- (xxx) If $3y = n$ and $2z + y = n$, then $y = z$, a contradiction since $y < z$.
- (xxxii) If $3z = n$ and $2z + y = n$, then $y = z$, a contradiction since $y < z$.
- (xxxiii) If $3y = n$ and $2y + z = n$, then $y = z$, a contradiction since $y < z$.
- (xxxiv) If $3z = n$ and $2y + z = n$, then $y = z$, a contradiction since $y < z$.
- (xxxv) If $y = 2x$ and $2x + y = n$, then $y = \frac{n}{2}$, a contradiction since $y < z \leq \frac{n}{2}$.
- (xxxvi) If $2y + z = n$ and $3x = n$, then $x = \frac{n}{3}$ and $y = \frac{n}{2} - \frac{z}{2}$. Given our assumption that $x < y$, it follows that $\frac{n}{2} - \frac{z}{2} > \frac{n}{3}$, and thus $z < \frac{n}{3}$. This is a contradiction since $x < y < z$.

- (xxxvii) If $2z + y = n$ and $2x + y = n$, then $x = z$, a contradiction since $x < z$.
- (xxxviii) If $z = x + y$ and $2x + y = n$, then $z = n - x$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $n - x \leq \frac{n}{2}$, and thus $x \geq \frac{n}{2}$. This is a contradiction since $x < z \leq \frac{n}{2}$.
- (xxxix) If $2y + x = n$ and $2x + y = n$, then $x = y$, a contradiction since $x < y$.
- (xl) If $z = 2y$ and $3x = n$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $2y \leq \frac{n}{2}$, and thus $y \leq \frac{n}{4}$. This is a contradiction since $x < y$.
- (xli) If $z = 2y$ and $2x + z = n$, then $y = \frac{n}{2} - x$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $2y \leq \frac{n}{2}$, and thus $y \leq \frac{n}{4}$. However, $y = \frac{n}{2} - x \leq \frac{n}{4}$ implies that $x \geq \frac{n}{4}$. This is a contradiction since $x < y$.
- (xlii) If $z = 2y$ and $2x + y = n$, then $y = n - 2x$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $2y \leq \frac{n}{2}$, and thus $y \leq \frac{n}{4}$. However, $y = n - 2x \leq \frac{n}{4}$ implies that $x \geq \frac{3n}{8}$. This is a contradiction since $x < y$.
- (xliii) If $z = 2y$ and $x + y + z = n$, then $y = \frac{n}{3} - \frac{x}{3}$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $2y \leq \frac{n}{2}$, and thus $y \leq \frac{n}{4}$. However, $y = \frac{n}{3} - \frac{x}{3} \leq \frac{n}{4}$ implies that $x \geq \frac{n}{4}$. This is a contradiction since $x < y$.
- (xliv) If $z = 2x$ and $2x + y = n$, then $z = n - y$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $n - y \leq \frac{n}{2}$, and thus $y \geq \frac{n}{2}$. This is a contradiction since $y < z \leq \frac{n}{2}$.
- (xlv) If $3x = n$ and $z = 2x$, then $x = \frac{n}{3}$ and $z = \frac{2n}{3}$, a contradiction since $z \leq \frac{n}{2}$.
- (xlvi) If $3z = n$ and $2y + x = n$, then $z = \frac{n}{3}$ and $y = \frac{n}{2} - \frac{x}{2}$. Given our assumption that $y < z$, it follows that $\frac{n}{2} - \frac{x}{2} < \frac{n}{3}$, and thus $x > \frac{n}{3}$. This is a contradiction since $x < z$.

- (xlvii) If $3z = n$ and $2x + y = n$, then $z = \frac{n}{3}$ and $y = n - 2x$. Given our assumption that $y < z$, it follows that $n - 2x < \frac{n}{3}$, and thus $x > \frac{n}{3}$. This is a contradiction since $x < z$.
- (xlviii) If $3x = n$ and $x + y + z = n$, then $x = \frac{n}{3}$ and $y = \frac{2n}{3} - z$. Given our assumption that $x < y$, it follows that $\frac{2n}{3} - z > \frac{n}{3}$, and thus $z < \frac{n}{3}$. This is a contradiction since $x < y < z$.
- (xlix) If $3z = n$ and $x + y + z = n$, then $z = \frac{n}{3}$ and $x = \frac{2n}{3} - y$. Given our assumption that $x < y$, it follows that $\frac{2n}{3} - y < y$, and thus $y > \frac{n}{3}$. This is a contradiction since $y < z$.
- (l) If $2x + z = n$ and $2x + y = n$, then $y = z$, a contradiction since $y < z$.
- (li) If $2x + y = n$ and $x + y + z = n$, then $x = z$, a contradiction since $x < z$.
- (lii) If $3x = n$ and $2z + y = n$, then $x = \frac{n}{3}$ and $y = n - 2z$. Given our assumption that $x < y$, it follows that $n - 2z > \frac{n}{3}$, and thus $z < \frac{n}{3}$. This is a contradiction since $x < z$.
- (liii) If $2x + y = n$ and $2y + z = n$, then $y = n - 2x$ and $z = n - 2y$. Given our assumption that $y < z$, it follows that $n - 2x < n - 2y$ thus implying that $x > y$. This is a contradiction since $x < y$.
- (liv) If $2x + z = n$ and $2z + y = n$, then $x = \frac{n}{2} - \frac{z}{2}$ and $y = n - 2z$. Given our assumption that $x < y$, it follows that $\frac{n}{2} - \frac{z}{2} < n - 2z$, and thus $z < \frac{n}{3}$. However, $z = \frac{n}{2} - \frac{y}{2} < \frac{n}{3}$ implies that $y > \frac{n}{3}$, a contradiction since $y < z$.
- (lv) If $2y + x = n$ and $2z + y = n$, then $x = n - 2y$ and $y = n - 2z$. Given our assumption that $x < y$, it follows that $n - 2y < n - 2z$ thus implying that $y > z$. This is a contradiction since $y < z$.

- (lvi) If $2z + x = n$ and $2x + y = n$, then $y = 4z - n$. Given our assumption that $y < z$, it follows that $4z - n < z$, and thus $z < \frac{n}{3}$. However, $z = \frac{n}{2} - \frac{x}{2} < \frac{n}{3}$ implies that $x > \frac{n}{3}$, a contradiction since $x < z$.
- (lvii) If $2x + z = n$ and $z = x + y$, then $x = \frac{n}{3} - \frac{y}{3}$. Given our assumption that $z \leq \frac{n}{2}$, it follows that $z = n - 2x \leq \frac{n}{2}$, and thus $x \geq \frac{n}{4}$. However, $x = \frac{n}{3} - \frac{y}{3} \geq \frac{n}{4}$ implies that $y \leq \frac{n}{4}$, a contradiction since $x < y$.
- (lviii) If $2z = n$, $2y + z = n$, and $z = 2x$, then $z = \frac{n}{2}$, $x = \frac{n}{4}$ and $y = \frac{n}{4}$, a contradiction since $x < y$.
- (lix) If $2z = n$, $2y + z = n$, and $z = x + y$, then $z = \frac{n}{2}$, $x = \frac{n}{4}$ and $y = \frac{n}{4}$, a contradiction since $x < y$.
- (lx) If $2z = n$, $z = 2x$ and $x + y + z = n$, then $z = \frac{n}{2}$, $x = \frac{n}{4}$ and $y = \frac{n}{4}$, a contradiction since $x < y$.
- (lxi) If $z = 2y$ and $2y + z = n$, then $y = \frac{n}{4}$ and $z = \frac{n}{2}$. However, $2z + x = n$ implies that $x = 0$, a contradiction since $x > 0$.
- (lxii) If $y = 2x$, $z = x + y$ and $3y = n$, then $y = \frac{n}{3}$, $x = \frac{n}{6}$ and $z = \frac{n}{2}$. However, $2z + x = n$ implies that $z = \frac{5n}{12}$, a contradiction since $z = \frac{n}{2}$.

■

Now we can determine which graphs in Class 13 (see Section 1.2) are well-covered.

Theorem 6.10 *Let $G = C(n, S)$ be a circulant graph on n vertices with a generating set $S = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} - A$, where $A \subseteq S$ and $|A| = 3$. Let $A = \{x, y, z\}$ where $0 < x < y < z \leq \frac{n}{2}$.*

(a) If n is divisible by two, then G is well-covered if and only if A is one of

$$\left\{ \frac{n}{6}, \frac{n}{3}, \frac{n}{2} \right\},$$

$$\left\{ x, \frac{n}{2} - x, \frac{n}{2} \right\}, \text{ or}$$

$$\left\{ x, y, \frac{n}{2} \right\}.$$

(b) If n is not divisible by two, then G is well-covered if and only if

either none of the following inequalities holds: $2z \neq n$, $3z \neq n$, $3y \neq n$, $3x \neq n$,

$2x + y \neq n$, $2y + x \neq n$, $x + y + z \neq n$, $2x + z \neq n$, $z \neq x + y$, $z \neq 2x$, $z \neq 2y$,

$2y + z \neq n$, $2z + y \neq n$, $2z + x \neq n$ and $y \neq 2x$; or

A is one of

$$(i) \left\{ \frac{n}{12}, \frac{n}{6}, \frac{n}{3} \right\},$$

$$(ii) \left\{ x, 2x, \frac{n}{3} \right\},$$

$$(iii) \left\{ x, \frac{n}{3} - x, \frac{n}{3} \right\},$$

$$(iv) \left\{ \frac{n}{6}, \frac{n}{3}, \frac{5n}{12} \right\},$$

$$(v) \left\{ x, \frac{n}{3}, 2x \right\},$$

$$(vi) \left\{ x, \frac{n}{3}, x + \frac{n}{3} \right\},$$

$$(vii) \left\{ x, \frac{n}{3}, \frac{2n}{3} - x \right\},$$

$$(viii) \left\{ x, \frac{n}{3}, n - 2x \right\},$$

$$(ix) \left\{ x, \frac{n}{3}, \frac{n}{2} - \frac{x}{2} \right\},$$

$$(x) \left\{ x, \frac{n}{2} - \frac{x}{2}, 2x \right\},$$

$$(xi) \left\{ x, \frac{n}{2} - \frac{x}{2}, n - 2x \right\},$$

$$(xii) \left\{ x, n - 3x, 2x \right\},$$

$$(xiii) \left\{ x, 2x, n - 3x \right\},$$

- (xiv) $\{x, y, n - x - y\}$,
- (xv) $\{\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}\}$,
- (xvi) $\{n - 3y, y, n - 2y\}$,
- (xvii) $\{\frac{n}{8}, \frac{n}{4}, \frac{3n}{8}\}$,
- (xviii) $\{3z - n, n - 2z, z\}$,
- (xix) $\{n - 2z, 3z - n, z\}$,
- (xx) $\{x, 2x, 3x\}$,
- (xxi) $\{x, y, x + y\}$,
- (xxii) $\{x, \frac{n}{2} - x, 2x\}$,
- (xxiii) $\{x, n - 4x, 2x\}$,
- (xxiv) $\{\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9}\}$,
- (xxv) $\{n - 4y, y, 2y\}$,
- (xxvi) $\{x, 2x, 4x\}$,
- (xxvii) $\{4y - n, y, n - 2y\}$,
- (xxviii) $\{x, 2x, n - 4x\}$,
- (xxix) $\{x, 2x, \frac{n}{2} - x\}$, *or*
- (xxx) $\{x, 2x, \frac{n}{2} - \frac{x}{2}\}$.

Furthermore,

- $\beta(G) = 2$ if either

$$z = \frac{n}{2}; \text{ or}$$

if all of $2z \neq n$, $3z \neq n$, $3y \neq n$, $3x \neq n$, $2x + y \neq n$, $2y + x \neq n$,
 $x + y + z \neq n$, $2x + z \neq n$, $z \neq x + y$, $z \neq 2x$, $z \neq 2y$, $2y + z \neq n$,
 $2z + y \neq n$, $2z + x \neq n$ and $y \neq 2x$ hold;

- $\beta(G) = 3$ if A is any one of

$$\left\{ \frac{n}{12}, \frac{n}{6}, \frac{n}{3} \right\},$$

$$\left\{ x, 2x, \frac{n}{3} \right\},$$

$$\left\{ x, \frac{n}{3} - x, \frac{n}{3} \right\},$$

$$\left\{ \frac{n}{6}, \frac{n}{3}, \frac{5n}{12} \right\},$$

$$\left\{ x, \frac{n}{3}, 2x \right\},$$

$$\left\{ x, \frac{n}{3}, x + \frac{n}{3} \right\},$$

$$\left\{ x, \frac{n}{3}, \frac{2n}{3} - x \right\},$$

$$\left\{ x, \frac{n}{3}, n - 2x \right\},$$

$$\left\{ x, \frac{n}{3}, \frac{n}{2} - \frac{x}{2} \right\},$$

$$\left\{ x, \frac{n}{2} - \frac{x}{2}, 2x \right\},$$

$$\left\{ x, \frac{n}{2} - \frac{x}{2}, n - 2x \right\},$$

$$\{x, y, n - x - y\},$$

$$\{x, y, x + y\};$$

$$\left\{ x, \frac{n}{2} - x, 2x \right\},$$

$$\{x, n - 4x, 2x\},$$

$$\left\{ \frac{n}{9}, \frac{2n}{9}, \frac{4n}{9} \right\},$$

$$\{n - 4y, y, 2y\},$$

$$\{x, 2x, 4x\},$$

$$\{4y - n, y, n - 2y\},$$

$$\{x, 2x, n - 4x\},$$

$$\{x, 2x, \frac{n}{2} - x\}, \text{ or}$$

$$\{x, 2x, \frac{n}{2} - \frac{x}{2}\};$$

- $\beta(G) = 4$ if A is any one of

$$\{x, \frac{n}{2} - x, \frac{n}{2}\};$$

$$\{x, n - 3x, 2x\},$$

$$\{x, 2x, n - 3x\},$$

$$\{n - 3y, y, n - 2y\},$$

$$\{\frac{n}{8}, \frac{n}{4}, \frac{3n}{8}\},$$

$$\{3z - n, n - 2z, z\},$$

$$\{n - 2z, 3z - n, z\}, \text{ or}$$

$$\{x, 2x, 3x\};$$

- $\beta(G) = 6$ if $A = \{\frac{n}{6}, \frac{n}{3}, \frac{n}{2}\}$; and

- $\beta(G) = 7$ if $A = \{\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}\}$.

Proof. Let $V(G) = \{v_i : i = 0, 1, \dots, n - 1\}$. By Corollary 6.2, $\overline{G} = C(n, \overline{S})$. Let $A = \{x, y, z\}$ and without loss of generality assume that $0 < x < y < z \leq \frac{n}{2}$. Note that $\overline{S} = A$. By Corollary 6.3, it suffices to show that all maximal cliques of \overline{G} containing v_0 are of the same size. Since v_0 is adjacent to v_{0+j} and v_{0-j} for each j in the set \overline{S} , $N_{\overline{G}}[v_0] = \{v_{-z}, v_{-y}, v_{-x}, v_x, v_y, v_z\}$.

Case 6.10.1 $2z = n$.

By Lemma 6.8, $v_z = v_{-z}$, and by Lemma 6.9, $3z \neq n$, $2z + x \neq n$ and $2z + y \neq n$. We consider two cases.

Case 6.10.1.1 $3x = n$.

By Lemma 6.8, $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$, $z \neq 2x$, $3y \neq n$, $2y + x \neq n$, $x + y + z \neq n$, $z \neq 2y$, $2x + y \neq n$, $2y + z \neq n$, $z \neq x + y$ and $2x + z \neq n$. Thus, we have one 3-vertex maximal clique and three 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{3}, y, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2 $3x \neq n$.

We consider two cases.

Case 6.10.1.2.1 $3y = n$.

By Lemma 6.8, $v_y \sim v_{-y}$, and by Lemma 6.9, $2y + z \neq n$, $z \neq 2y$, $2y + x \neq n$ and $2x + y \neq n$. We consider two cases.

Case 6.10.1.2.1.1 $x + y + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$, $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$, and by Lemma 6.9, $2x + z \neq n$. Observe that $2z = n$ together with $x + y + z = n$ implies that $z = x + y$, and hence by Lemma 6.9, $z \neq 2x$, and by Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$ and $v_{-x} \sim v_{-z}$. We also note that $2z = n$ together with $3y = n$ and $z = x + y$ implies that $y = 2x$, and hence by Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly one 6-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{6}, \frac{n}{3}, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 6$.

Case 6.10.1.2.1.2 $x + y + z \neq n$.

Observe that $2z = n$ together with $x + y + z \neq n$ implies that $z \neq x + y$, and $2z = n$ together with $3y = n$ and $z \neq x + y$ implies that $y \neq 2x$. We consider two cases.

Case 6.10.1.2.1.2.1 $2x + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_x \sim v_{-x}$. Observe that $2z = n$ together with $2x + z = n$ implies that $z = 2x$, and hence by Lemma 6.8, $v_x \sim v_z$ and $v_{-x} \sim v_{-z}$. Thus, we have one 4-vertex maximal clique and one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{4}, \frac{n}{3}, \frac{n}{2} \right\}$ and G is not well-covered.

Case 6.10.1.2.1.2.2 $2x + z \neq n$.

Observe that $2z = n$ together with $2x + z \neq n$ implies that $z \neq 2x$. Thus, we have one 3-vertex maximal clique and three 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \left\{ x, \frac{n}{3}, \frac{n}{2} \right\}$ and G is not well-covered.

Case 6.10.1.2.2 $3y \neq n$.

We consider two cases.

Case 6.10.1.2.2.1 $2y + z = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$ and $v_y \sim v_{-y}$, and by Lemma 6.9, $2x + y \neq n$, $2y + x \neq n$, $2x + z \neq n$ and $x + y + z \neq n$. Observe that $2z = n$ together with $2y + z = n$ implies that $z = 2y$, and hence by Lemma 6.9, $z \neq 2x$, and by Lemma 6.8, $v_y \sim v_z$ and $v_{-y} \sim v_{-z}$. We also note that $2z = n$ together with $x + y + z \neq n$ implies that $z \neq x + y$, and $2z = n$ together with $2x + z \neq n$ implies that $z \neq 2x$. We consider two cases.

Case 6.10.1.2.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have three 3-vertex maximal cliques and one 4-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{8}, \frac{n}{4}, \frac{n}{2} \right\}$ and G is not well-covered.

Case 6.10.1.2.2.1.2 $y \neq 2x$.

We have one 4-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \left\{ x, \frac{n}{4}, \frac{n}{2} \right\}$ and G is not well-covered.

Case 6.10.1.2.2.2 $2y + z \neq n$.

We consider two cases.

Case 6.10.1.2.2.2.1 $x + y + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$, $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$, and by Lemma 6.9, $2x + z \neq n$, $2x + y \neq n$, $z \neq 2y$ and $2y + x \neq n$. Observe that $2z = n$ together with $x + y + z = n$ implies that $z = x + y$, and hence by Lemma 6.9, $z \neq 2x$, and by Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$ and $v_{-x} \sim v_{-z}$. We consider two cases.

Case 6.10.1.2.2.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Observe that $2z = n$ together with $y = 2x$ and $z = x + y$ implies that $3y = n$, and hence by Lemma 6.8, $v_y \sim v_{-y}$. Thus, we have exactly one 6-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{6}, \frac{n}{3}, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 6$.

Case 6.10.1.2.2.2.1.2 $y \neq 2x$.

We have exactly two 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{2} - x, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.1.2.2.2.2 $x + y + z \neq n$.

Observe that $2z = n$ together with $x + y + z \neq n$ implies that $z \neq x + y$. We consider two cases.

Case 6.10.1.2.2.2.2.1 $2x + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $2x + y \neq n$, $z \neq 2y$ and $y \neq 2x$. Observe that $2z = n$ together with $2x + z = n$ implies that $z = 2x$, and hence by Lemma 6.8, $v_x \sim v_z$ and $v_{-x} \sim v_{-z}$. We consider two cases.

Case 6.10.1.2.2.2.1.1 $2y + x = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$ and $v_y \sim v_{-y}$. Thus, we have one 4-vertex maximal clique and three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{4}, \frac{3n}{8}, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.1.2 $2y + x \neq n$.

We have one 4-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{4}, y, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2 $2x + z \neq n$.

Observe that $2z = n$ together with $2x + z \neq n$ implies that $z \neq 2x$. We consider two cases.

Case 6.10.1.2.2.2.2.1 $2y + x = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$ and $v_y \sim v_{-y}$, and by Lemma 6.9, $2x + y \neq n$ and $z \neq 2y$. We consider two cases.

Case 6.10.1.2.2.2.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have one 5-vertex maximal clique and one 2-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{5}, \frac{2n}{5}, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.1.2 $y \neq 2x$.

We have three 3-vertex maximal cliques and one 2-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{n - 2y, y, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.2 $2y + x \neq n$.

We consider two cases.

Case 6.10.1.2.2.2.2.2.1 $2x + y = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $z \neq 2y$ and $y \neq 2x$. Hence, we have three 3-vertex maximal cliques and one 2-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{x, n - 2x, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.2.2 $2x + y \neq n$.

We consider two cases.

Case 6.10.1.2.2.2.2.2.2.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. We consider two cases.

Case 6.10.1.2.2.2.2.2.2.1.1 $z = 2y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$ and $v_y \sim v_{-y}$. Thus, we have three 3-vertex maximal cliques and one 4-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{8}, \frac{n}{4}, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.2.2.1.2 $z \neq 2y$.

We have one 2-vertex maximal clique and three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.2.2.2 $y \neq 2x$.

We consider two cases.

Case 6.10.1.2.2.2.2.2.2.2.1 $z = 2y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$ and $v_y \sim v_{-y}$. Thus, we have one 4-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{4}, \frac{n}{2}\}$ and G is not well-covered.

Case 6.10.1.2.2.2.2.2.2.2.2 $z \neq 2y$.

We have exactly five 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, y, \frac{n}{2}\}$, G is well-covered and $\beta(G) = 2$.

Case 6.10.2 $2z \neq n$.

We consider two cases.

Case 6.10.2.1 $3z = n$.

By Lemma 6.8, $v_z \sim v_{-z}$, and by Lemma 6.9, $2y + x \neq n$, $3x \neq n$, $3y \neq n$, $2x + z \neq n$, $2y + z \neq n$, $2x + y \neq n$, $x + y + z \neq n$, $2z + y \neq n$, and $2z + x \neq n$. We consider two cases.

Case 6.10.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $z \neq 2x$. We consider two cases.

Case 6.10.2.1.1.1 $z = 2y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$ and $v_y \sim v_{-y}$, and by Lemma 6.9, $z \neq x + y$. Thus, we have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{12}, \frac{n}{6}, \frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.1.1.2 $z \neq 2y$.

We consider two cases.

Case 6.10.2.1.1.2.1 $z = x + y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$, $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$. Thus, we have four 4-vertex maximal cliques and one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{9}, \frac{2n}{9}, \frac{n}{3}\}$ and G is not well-covered.

Case 6.10.2.1.1.2.2 $z \neq x + y$.

We have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x, \frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.1.2 $y \neq 2x$.

We consider two cases.

Case 6.10.2.1.2.1 $z = 2x$.

By Lemma 6.8, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $z \neq x + y$ and $z \neq 2y$. Thus, we have four 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{6}, y, \frac{n}{3}\}$ and G is not well-covered.

Case 6.10.2.1.2.2 $z \neq 2x$.

We consider two cases.

Case 6.10.2.1.2.2.1 $z = 2y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$ and $v_y \sim v_{-y}$, by Lemma 6.9, $z \neq x + y$. Thus, we have four 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{6}, \frac{n}{3}\}$ and G is not well-covered.

Case 6.10.2.1.2.2.2 $z \neq 2y$.

We consider two cases.

Case 6.10.2.1.2.2.2.1 $z = x + y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$, $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$. Thus, we have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3} - x, \frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.1.2.2.2.2 $z \neq x + y$.

We have one 3-vertex maximal clique and four 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, y, \frac{n}{3}\}$ and G is not well-covered.

Case 6.10.2.2 $3z \neq n$.

We consider two cases.

Case 6.10.2.2.1 $3y = n$.

By Lemma 6.8, $v_y \sim v_{-y}$, and by Lemma 6.9, $z \neq 2y$, $2y + x \neq n$, $3x \neq n$, $2x + y \neq n$, $2y + z \neq n$ and $2z + y \neq n$. We consider two cases.

Case 6.10.2.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $z \neq 2x$ and $2x + z \neq n$. Observe that $z = x + y$ together with $3y = n$ and $y = 2x$ is inconsistent with our assumption that $2z \neq n$, and hence $z \neq x + y$. We also note that $x + y + z = n$ together with $3y = n$ and $y = 2x$ is inconsistent with our assumption that $2z \neq n$, and hence $x + y + z \neq n$. We consider two cases.

Case 6.10.2.2.1.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Thus, we have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{6}, \frac{n}{3}, \frac{5n}{12} \right\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.1.2 $2z + x \neq n$.

We have four 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{6}, \frac{n}{3}, z \right\}$ and G is not well-covered.

Case 6.10.2.2.1.2 $y \neq 2x$.

We consider two cases.

Case 6.10.2.2.1.2.1 $z = 2x$.

By Lemma 6.8, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $z \neq x + y$. Observe that $2x + z = n$ together with $z = 2x$ is inconsistent with our assumption that $2z \neq n$, and hence $2x + z \neq n$. We consider two cases.

Case 6.10.2.2.1.2.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$, and by Lemma 6.9, $x + y + z \neq n$. Thus, we have one 5-vertex maximal clique and one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{5}, \frac{n}{3}, \frac{2n}{5} \right\}$ and G is not well-covered.

Case 6.10.2.2.1.2.1.2 $2z + x \neq n$.

We consider two cases.

Case 6.10.2.2.1.2.1.2.1 $x + y + z = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$, $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$. Thus, we have four 4-vertex maximal cliques and one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{2n}{9}, \frac{n}{3}, \frac{4n}{9} \right\}$ and G is not well-covered.

Case 6.10.2.2.1.2.1.2.2 $x + y + z \neq n$.

We have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \left\{ x, \frac{n}{3}, 2x \right\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.2.2 $z \neq 2x$.

We consider two cases.

Case 6.10.2.2.1.2.2.1 $z = x + y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$, $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$, and by Lemma 6.9, $2x + z \neq n$. Observe that $x + y + z = n$ together with $z = x + y$ is inconsistent with our assumption that $2z \neq n$, and hence $x + y + z \neq n$. We consider two cases.

Case 6.10.2.2.1.2.2.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Thus, we have four 4-vertex maximal cliques and one 3-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \left\{ \frac{n}{9}, \frac{n}{3}, \frac{4n}{9} \right\}$ and G is not well-covered.

Case 6.10.2.2.1.2.2.1.2 $2z + x \neq n$.

We have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}, x + \frac{n}{3}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.2.2.2 $z \neq x + y$.

We consider two cases.

Case 6.10.2.2.1.2.2.2.1 $x + y + z = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$, $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$, and by Lemma 6.9, $2x + z \neq n$ and $2z + x \neq n$. We have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}, \frac{2n}{3} - x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.2.2.2.2 $x + y + z \neq n$.

We consider two cases.

Case 6.10.2.2.1.2.2.2.2.1 $2x + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $2z + x \neq n$. Thus, we have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}, n - 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.2.2.2.2.2 $2x + z \neq n$.

We consider two cases.

Case 6.10.2.2.1.2.2.2.2.2.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$, and by Lemma 6.9, $x + y + z \neq n$. Thus, we have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}, \frac{n}{2} - \frac{x}{2}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.1.2.2.2.2.2 $2z + x \neq n$.

We have one 3-vertex maximal clique and four 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{3}, z\}$ and G is not well-covered.

Case 6.10.2.2.2 $3y \neq n$.

We consider two cases.

Case 6.10.2.2.2.1 $3x = n$.

By Lemma 6.8, $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$, $z \neq 2x$, $2y + x \neq n$, $x + y + z \neq n$, $z \neq 2y$, $2x + y \neq n$, $2y + z \neq n$, $z \neq x + y$, $2x + z \neq n$, $2z + y \neq n$ and $2z + x \neq n$. Thus, we have one 3-vertex maximal clique and four 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{3}, y, z\}$ and G is not well-covered.

Case 6.10.2.2.2.2 $3x \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.1 $2x + y = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$, $z \neq x + y$, $2y + x \neq n$, $z \neq 2y$, $z \neq 2x$, $2x + z \neq n$, $x + y + z \neq n$, $2y + z \neq n$, $2z + y \neq n$ and $2z + x \neq n$. Thus, we have three 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, n - 2x, z\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2 $2x + y \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.1 $2y + x = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$ and $v_y \sim v_{-y}$, and by Lemma 6.9, $z \neq x + y$, $x + y + z \neq n$, $2z + x \neq n$, $2y + z \neq n$, $2z + y \neq n$ and $z \neq 2y$. We consider two cases.

Case 6.10.2.2.2.2.1.1 $z = 2x$.

By Lemma 6.8, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$. Observe that $2x + z = n$ together with $z = 2x$ is inconsistent with our assumption that $2z \neq n$, and hence $2x + z \neq n$. Thus, we have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{2} - \frac{x}{2}, 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.1.2 $z \neq 2x$.

We consider two cases.

Case 6.10.2.2.2.2.1.2.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $2x + z \neq n$. Thus, we have one 5-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{5}, \frac{2n}{5}, z\}$ and G is not well-covered.

Case 6.10.2.2.2.2.1.2.2 $y \neq 2x$.

We consider two cases.

Case 6.10.2.2.2.2.1.2.2.1 $2x + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_x \sim v_{-x}$. Thus, we have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{2} - \frac{x}{2}, n - 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.1.2.2.2 $2x + z \neq n$.

We have three 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{2} - \frac{x}{2}, z\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2 $2y + x \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.1 $x + y + z = n$.

By Lemma 6.8, $v_x \sim v_{-y}$, $v_{-x} \sim v_y$, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$, $v_y \sim v_{-z}$ and $v_{-y} \sim v_z$, and by Lemma 6.9, $2x + z \neq n$, $2y + z \neq n$, $z \neq 2y$, $2z + y \neq n$ and $2z + x \neq n$. Observe that $x + y + z = n$ together with $z = x + y$ is inconsistent with our assumption that $2z \neq n$, and hence $z \neq x + y$. We consider two cases.

Case 6.10.2.2.2.2.2.1.1 $z = 2x$.

By Lemma 6.8, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$. Thus, we have exactly two 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, n - 3x, 2x\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.1.2 $z \neq 2x$.

We consider two cases.

Case 6.10.2.2.2.2.2.1.2.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly four 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x, n - 3x\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.1.2.2 $y \neq 2x$.

We have exactly four 3-vertex maximal cliques in \overline{G} containing v_0 . Therefore, $A = \{x, y, n - x - y\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2 $x + y + z \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.1 $2x + z = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$, $z \neq 2y$, $2y + z \neq n$, $z \neq x + y$, $2z + y \neq n$ and $2z + x \neq n$. Observe that $2x + z = n$

together with $z = 2x$ is inconsistent with our assumption that $2z \neq n$, and hence $z \neq 2x$. Thus, we have three 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, y, n - 2x\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.2 $2x + z \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1 $z = x + y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$, $v_x \sim v_{-y}$ and $v_{-x} \sim v_y$, and by Lemma 6.9, $z \neq 2y$ and $z \neq 2x$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1.1 $2y + z = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$ and $v_y \sim v_{-y}$, by Lemma 6.9, $2z + y \neq n$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Observe that $2y + z = n$ together with $z = x + y$ and $2z + x = n$ implies that $y = 2x$, and hence by Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly one 7-vertex maximal clique in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}\}$, G is well-covered and $\beta(G) = 7$.

Case 6.10.2.2.2.2.2.2.2.2.1.1.2 $2z + x \neq n$.

Observe that $2y + z = n$ together with $2z + x \neq n$ and $z = x + y$ implies that $y \neq 2x$. Thus, we have exactly four 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{n - 3y, y, n - 2y\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.2.2.2.1.2 $2y + z \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.1.2.1 $2z + y = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$, and $v_z \sim v_{-z}$, and by Lemma 6.9, $2z + x \neq n$. We consider two cases.

Case 6.10.2.2.2.2.2.2.1.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly six 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{8}, \frac{n}{4}, \frac{3n}{8}\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.2.1.2.1.2 $y \neq 2x$.

Thus, we have exactly four 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{3z - n, n - 2z, z\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.2.1.2.2 $2z + y \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.1.2.2.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Observe that $2y + z \neq n$ together with $2z + x = n$ and $z = x + y$ implies that $y \neq 2x$. Thus, we have exactly four 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{n - 2z, 3z - n, z\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.2.1.2.2.2 $2z + x \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.1.2.2.2.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly four 4-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x, 3x\}$, G is well-covered and $\beta(G) = 4$.

Case 6.10.2.2.2.2.2.2.2.1.2.2.2 $y \neq 2x$.

We have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Therefore, $A = \{x, y, x + y\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.2 $z \neq x + y$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1 $z = 2x$.

By Lemma 6.8, $v_x \sim v_z$, $v_{-x} \sim v_{-z}$ and $v_x \sim v_{-x}$, and by Lemma 6.9, $y \neq 2x$ and $z \neq 2y$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1.1 $2y + z = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$ and $v_y \sim v_{-y}$, and by Lemma 6.9, $2z + y \neq n$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Thus, we have one 5-vertex maximal clique and two 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{5}, \frac{3n}{10}, \frac{2n}{5}\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.1.1.2 $2z + x \neq n$.

We have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Therefore, $A = \{x, \frac{n}{2} - x, 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.1.2 $2y + z \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1.2.1 $2z + y = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$, and $v_z \sim v_{-z}$ and by Lemma 6.9, $2z + x \neq n$. Thus, we have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, n - 4x, 2x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.1.2.2 $2z + y \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1.2.2.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$. Thus, we have one 5-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{5}, y, \frac{2n}{5}\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.1.2.2.2 $2z + x \neq n$.

We have three 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, y, 2x\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.2 $z \neq 2x$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1 $z = 2y$.

By Lemma 6.8, $v_y \sim v_z$, $v_{-y} \sim v_{-z}$ and $v_y \sim v_{-y}$. Observe that $2y + z = n$ together with $z = 2y$ is inconsistent with our assumption that $2z \neq n$, and hence $2y + z \neq n$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1.1 $2z + x = n$.

By Lemma 6.8, $v_x \sim v_{-z}$, $v_{-x} \sim v_z$ and $v_z \sim v_{-z}$, and by Lemma 6.9, $2z + y \neq n$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.1.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly seven 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{9}, \frac{2n}{9}, \frac{4n}{9}\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.2.1.1.2 $y \neq 2x$.

We have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{n - 4y, y, 2y\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.2.1.2 $2z + x \neq n$.

We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1.2.1 $2z + y = n$.

By Lemma 6.8, $v_y \sim v_{-z}$, $v_{-y} \sim v_z$, and $v_z \sim v_{-z}$. We consider two cases.

Case 6.10.2.2.2.2.2.2.2.2.1.2.1.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have one 5-vertex maximal clique and three 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{\frac{n}{10}, \frac{n}{5}, \frac{2n}{5}\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.2.1.2.1.2 $y \neq 2x$.

We have one 5-vertex maximal clique and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, \frac{n}{5}, \frac{2n}{5}\}$ and G is not well-covered.

Case 6.10.2.2.2.2.2.2.2.2.1.2.2 $2z + y \neq n$.

Case 6.10.2.2.2.2.2.2.2.2.1.2.2.1 $y = 2x$.

By Lemma 6.8, $v_x \sim v_y$, $v_{-x} \sim v_{-y}$ and $v_x \sim v_{-x}$. Thus, we have exactly six 3-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, 2x, 4x\}$, G is well-covered and $\beta(G) = 3$.

Case 6.10.2.2.2.2.2.2.2.2.1.2.2.2 $y \neq 2x$.

We have three 3-vertex maximal cliques and two 2-vertex maximal cliques in \overline{G} containing v_0 . Hence, $A = \{x, y, 2y\}$ and G is not well-covered.

Chapter 7

Conclusion

In this thesis, we investigated circulant graphs. The main purpose of our research was to determine necessary and sufficient conditions for certain families of circulant graphs to be well-covered. Since the problem of determining well-coveredness of an arbitrary circulant graph is co-NP-complete, attaining full characterization will be difficult. We were able to identify several families of well-covered circulant graphs that can be identified in polynomial time. These families could be extended by applying the lexicographic product.

Future research in this area could investigate

- (i) A well-covered graph is 1-well-covered if $G - v$ is well-covered for all v in the vertex set of G . Can we find necessary and sufficient conditions for a circulant graph to be 1-well-covered?
- (ii) The girth of a graph with a cycle is the length of its shortest cycle while a graph with no cycle has infinite girth. Can we characterize well-covered circulants of girth three and four?

Bibliography

- [1] A. Finbow and B. Hartnell, A game related to covering by stars, *Ars Combin.*, **16-A** (1983) 189–198.
- [2] A. Finbow, B. Hartnell and R. J. Nowakowski, A characterization of well-covered graphs of girth 5 or greater, *J. Combin. Theory Ser. B*, **57** (1993) 44–68.
- [3] A. Finbow, B. Hartnell and R. J. Nowakowski, A characterization of well-covered graphs that contain neither 4- nor 5-cycles, *J. Graph Theory*, **18** (1994) 713–721.
- [4] A. Finbow and B. Hartnell, A characterization of parity graphs containing no cycle of order five or less, *Ars Combin.*, **40** (1995) 227–234.
- [5] A. Finbow and B. Hartnell, On 1-well-covered graphs, *RMS-Lecture Notes Series*, **13** (2010) 33–44.
- [6] B. Hartnell and M. D. Plummer, On 4-connected claw-free well-covered graphs, *Discrete Applied Math.*, **64** (1996) 57–65.
- [7] B. L. Hartnell, On the local structure of well-covered graphs without 4-cycles, *Ars Combin.*, **45** (1997) 77–86.
- [8] B. L. Hartnell, Well-covered graphs, *J. Combin. Math. Combin. Comput.*, **29** (1999) 107–115.
- [9] B. L. Hartnell, A characterization of the 1-well-covered graphs with no 4-cycles, *Graph theory in Paris*, *Trends Math.*, (2007) 219–224.
- [10] C. A. Whitehead, A characterization of well-covered claw-free graphs containing no 4-cycles, *Ars Combin.*, **39** (1995) 189–198.
- [11] C. H. Li, On isomorphisms of finite Cayley graphs - a survey, *Discrete Math.*, **256** (2002) 301–334.
- [12] D. Tankus and M. Tarsi, Well-covered claw-free graphs, *J. Combin. Theory Ser. B*, **66** (1996) 293–302.
- [13] D. Tankus and M. Tarsi, The structure of well-covered graphs and the complexity of their recognition problems, *J. Combin. Theory Ser. B*, **69** (1997) 230–233.
- [14] E. L. C. King, Characterizing a subclass of well-covered graphs, *Congressus Numerantium*, **160** (2003) 7–31.
- [15] G. Chartrand and P. Zhang, “Introduction to Graph Theory”, Boston: McGraw-Hill Higher Education, (2005).

- [16] G. Ravindra, Well-covered graphs, *J. Combin. Inform. System Sci.*, **2** (1977) 20–21.
- [17] I. Broere and J. H. Hattingh, Products of circulant graphs, *Quaestiones Math.*, **13** (1990) 191–216.
- [18] J. A. Bondy and U. S. R. Murty, “Graph Theory With Applications”, North Holland, New York, (1976).
- [19] J. A. W. Staples, On some subclasses of well-covered graphs, Ph.D. thesis, Vanderbilt University, (1975).
- [20] J. A. W. Staples, On some subclasses of well-covered graphs, *J. Graph Theory*, **3** (1979) 197–204.
- [21] J. Gross and J. Yellen, “Handbook of Graph Theory”, Boca Raton: CRC Press, (2004).
- [22] J. I. Brown, R. J. Nowakowski and I. E. Zverovich, The structure of well-covered graphs with no cycles of length 4, *Discrete Math.*, **307** (2007) 2235–2245.
- [23] J. Topp and L. Volkmann, On the well-coveredness of products of graphs, *Ars Combin.*, **33** (1992) 199–215.
- [24] M. Muzychuk, A solution of the isomorphism problem for circulant graphs, *Proc. London Math. Soc.*, **88** (2004) 1–41.
- [25] M. Pinter, W_2 graphs and strongly well-covered graphs: two well-covered graph subclasses, Ph.D. thesis, Vanderbilt University, (1991).
- [26] M. Pinter, Planar regular one-well-covered graphs, *Cong. Numer.*, **91** (1992) 159–187.
- [27] M. Pinter, A class of planar well-covered graphs with girth four, *J. Graph Theory*, **19** (1995) 69–81.
- [28] M. Pinter, A class of well-covered graphs with girth four, *Ars Combin.*, **45** (1997) 241–255.
- [29] M. D. Plummer, Some covering concepts in graphs, *J. Combin. Theory*, **8** (1970) 91–98.
- [30] M. D. Plummer, Well-covered graphs: a survey, *Quaestiones Mathematicae*, **16** (1993) 253–287.
- [31] O. Favaron, Very well-covered graphs, *Discrete Math.*, **42** (1982) 177–187.
- [32] A. O. Fradkin, On the well-coveredness of Cartesian products of graphs, *Discrete Math.*, **309** (2009) 238–246.

- [33] R. Hoshino, Independence polynomials of circulant graphs, Ph.D. thesis, Dalhousie University, (2008).
- [34] R. Hoshino and J. I. Brown, Well-covered circulant graphs, *Discrete Math.*, **311** (2011) 244–251.
- [35] R. S. Sankaranarayana and L. K. Stewart, Complexity results for well-covered graphs, *Networks*, **22** (1992) 247–262.
- [36] R. S. Sankaranarayana, Well-covered graphs: some new sub-classes and complexity results, Ph.D. thesis, University of Alberta, (1994).
- [37] R. S. Sankaranarayana and L. K. Stewart, Recursively decomposable well-covered graphs, *Discrete Math.*, **161** (1996) 243–263.
- [38] S. L. Gasquoin, B. L. Hartnell, R. J. Nowakowski and C. A. Whitehead, Techniques for constructing well-covered graphs with no 4-cycles, *J. Combin. Math. Combin. Comput.*, **17** (1995) 65–87.
- [39] S. R. Campbell, Some results on cubic well-covered graphs, Ph.D. thesis, Vanderbilt University, (1987).
- [40] S. R. Campbell and M. D. Plummer, On well-covered 3-polytopes, *Ars Combin.*, **25-A** (1988) 215–242.
- [41] S. R. Campbell, M. N. Ellingham and G. F. Royle, A characterization of well-covered cubic graphs, *J. Combin. Math. Combin. Comput.*, **13** (1993) 193–212.
- [42] T. J. Gionet, E. L. C. King and Y. Sha, A revision and extension of results on 4-regular, 4-connected, claw-free graphs, *Discrete Applied Math.*, **159** (2011) 1225–1230.
- [43] V. Chvátal and P. J. Slater, A note on well-covered graphs, Quo Vadis, Graph Theory?, *Annals Discrete Math.*, **55** (1993) 179–182.
- [44] Y. Caro, A. Sebö and M. Tarsi, Recognizing greedy structures, *J. Algorithms*, **20(1)** (1996) 137–156.
- [45] Y. Caro, Subdivisions, parity and well-covered graphs, *J. Graph Theory*, **25** (1997) 85–94.
- [46] Y. Caro and B. Hartnell, A characterization of Z_m -well-covered graphs of girth 6 or more, *J. Graph Theory*, **33** (2000) 246–255.